Economics 200B Prof. R. Starr UCSD Winter 2010

Lecture Notes for January 6, 2010; part 2

Integrating Production and Multiple Consumption Decisions: A $2 \times 2 \times 2$ Model

Competitive Equilibrium:

- Production and consumption plans are each separately optimized at the prevailing prices – the same prices facing all firms and households.
- Markets clear; supply equals demand. There is a single consistent point chosen in the Edgeworth box. The dimensions of the box are set to reflect the production decision. Consumption decisions are consistent with one another and with the output produced, precisely exhausting available goods.

Pareto Efficiency:

- The production sector is technically efficient; each firm is producing maximal output from its inputs and there is no reallocation of inputs among firms that would result in a higher output of some goods without a reduction in output of others.
- The MRS, the tradeoff in consumption between goods, is the same for all households. Equating MRSs results in locating the consumption plan at a point on the contract curve.
- The MRS equals the MRT: The tradeoff in output choice is the same on both the production and consumption sides. The budget line in the Edgeworth box and the

isoprofit line tangent to the production choice have the same slope equal to the price ratio (times -1).

First Fundamental Theorem of Welfare Economics: A competitive equilibrium allocation is Pareto efficient. This is ensured by separate firm and household optimizations facing common prices.

For convenience, this treatment will concentrate on interior solutions (no corner solutions) .

A $2 \times 2 \times 2$ Model

Two factors: land denoted T, and labor denoted L.

Two goods, x and y.

Two households, 1 and 2.

1's endowment of L is L^1, 2's is L^2. $L^1 + L^2 = L^o$.

1's endowment of T is T^1, 2's is T^2. $T^1 + T^2 = T^o$.

The prevailing wage rate of labor is w, and the rental rate on land is r.

$f(L^x, T^x) = x$, where L^x is L used to produce x, T^x is T used to produce x. $f(L^x, T^x) \geq 0$ for $L^x \geq 0$, $T^x \geq 0$; $f(0, 0) = 0$.

$g(L^y, T^y) = y$ where L^y is L used to produce y, T^y is T used to produce y. $g(L^y, T^y) \geq 0$ for $L^y \geq 0$, $T^y \geq 0$; $g(0, 0) = 0$.

p^x , p^y.

$\Pi^x = p^x f(L^x, T^x) - wL^x - rT^x$. $\Pi^y = p^y g(L^y, T^y) - wL^y - rT^y$.

α^{1x}, α^{1y}. α^{2x}, α^{2y}. $\alpha^{1x} + \alpha^{2x} = 1$. $\alpha^{1y} + \alpha^{2y} = 1$.

$I^1 = wL^1 + rT^1 + \alpha^{1x}\Pi^x + \alpha^{1y}\Pi^y$

$I^2 = wL^2 + rT^2 + \alpha^{2x}\Pi^x + \alpha^{2y}\Pi^y$

$u^1(x^1, y^1)$. $u^2(x^2, y^2)$.

Assume f, g, u^1, u^2, to be strictly concave, differentiable. Assume all solutions are interior. Subscripts denote partial derivatives.

Technical Efficiency

The production possibility set can be described as

$PPS = \{(x, y)|x, y \geq 0; T^x, T^y, L^x, L^y \geq 0; T^x + T^y \leq T^o; L^x + L^y \leq L^o; x \leq f(L^x, T^y); y \leq g(L^y, T^y)\}$.

It is described as the set: $E = \{(x', y')|x' = f(L^x, T^x), y' = g(L^y, T^y)$,

(1)$L^x + L^y = L^o; L^x, L^y \geq 0;$

(2)$T^x + T^y = T^o; T^x, T^y \geq 0$

L^x, L^y, T^x, T^y are chosen to maximize $f(L^x, T^x)$ subject to (1), (2) and subject to $g(L^y, T^y)\} = y^o$ for arbitrary attainable y^o.

$Q = f(L^x, T^x) - \lambda(g(L^o - L^x, T^o - T^x) - y^o)$.

First order conditions for optimizing Q are $\frac{\partial \mathrm{Q}}{\partial \mathrm{L}^{\mathrm{x}}} = f_L + \lambda g_L = 0$

(where the subscripts refer to partial derivatives)

$$\frac{\partial \mathrm{Q}}{\partial \mathrm{T}^{\mathrm{x}}} = f_T + \lambda g_T = 0$$

Rearranging terms we have

$$\frac{\partial \mathrm{f}}{\partial \mathrm{L}}/\frac{\partial \mathrm{f}}{\partial \mathrm{T}} = f_L/f_T = \frac{\partial \mathrm{g}}{\partial \mathrm{L}}/\frac{\partial \mathrm{g}}{\partial \mathrm{T}} = g_L/g_T = \frac{\mathrm{dT}^{\mathrm{x}}}{\mathrm{dL}^{\mathrm{x}}}|_{f=constant} = \frac{\mathrm{dT}^{\mathrm{y}}}{\mathrm{dL}^{\mathrm{y}}}|_{g=constant}.$$

Pareto Efficiency Definition (Pareto efficiency): $x^{*1}, y^{*1}, x^{*2}, y^{*2}$ is said to be Pareto efficient if it is attainable (an element of PPS) and if (x^{*1}, y^{*1}) maximizes $u^1(x, y)$ subject to (1), (2), and

(3)$x^1 + x^2 = f(L^x, T^x)$, and

(4)$y^1 + y^2 = g(L^y, T^y)$,

and subject to $u(x^2, y^2) = u^2(x^{*2}, y^{*2})$.

To describe a Pareto efficient allocation, we will ask how to maximize $u^1(x^1, y^1)$ subject to resource and technology constraints and subject to a fixed level of $u^2(x^2, y^2)$. That is we want to choose $x^1, y^1, L^x, L^y, T^x, T^y$, to maximize $u^1(x^1, y^1)$ subject to $u^2(x^2, y^2) = u^{2o}$ and subject to $L^x + L^y = L^o$, $T^x + T^y = T^o$, $x^1 + x^2 = f(L^x, T^x)$, $y^1 + y^2 = g(L^y, T^y)$.

This problem leads to the (world's biggest) Lagrangian,

$$H = u^1(x^1, y^1) + \lambda[u^2(x^2, y^2) - u^{2o}] + \mu[y^1 + y^2 - g(L^y, T^y)]$$

$$+\nu[x^1 + x^2 - f(L^x, T^x)] + \eta[T^x + T^y - T^o] + \epsilon[L^x + L^y - L^o]$$

First order conditions for the Lagrangian are

$\frac{\partial \mathrm{H}}{\partial \mathrm{x}^1} = u^1_x + \nu = 0,$
$\frac{\partial \mathrm{H}}{\partial \mathrm{y}^1} = u^1_y + \mu = 0$
$\frac{\partial \mathrm{H}}{\partial \mathrm{x}^2} = \lambda u^2_x + \nu = 0,$
$\frac{\partial \mathrm{H}}{\partial \mathrm{y}^2} = \lambda u^2_y + \mu = 0$
$\frac{\partial \mathrm{H}}{\partial \mathrm{L}^\mathrm{x}} = -\nu f_L - \epsilon = 0$,
$\frac{\partial \mathrm{H}}{\partial \mathrm{L}^\mathrm{y}} = -\mu g_L - \epsilon = 0$
$\frac{\partial \mathrm{H}}{\partial \mathrm{T}^\mathrm{x}} = -\nu f_T - \eta = 0,$
$\frac{\partial \mathrm{H}}{\partial \mathrm{T}^\mathrm{y}} = -\mu g_T - \eta = 0.$

The first order conditions for the Lagrangian lead to
$u^1_x = -\nu$, $u^1_y = -\mu$
$u^2_x = -\nu/\lambda$, $u^2_y = -\mu/\lambda$,
which imply
$\frac{\mathrm{u}^1_\mathrm{x}}{\mathrm{u}^1_\mathrm{y}} = \frac{\nu}{\mu} = \frac{\mathrm{u}^2_\mathrm{x}}{\mathrm{u}^2_\mathrm{y}}$
demonstrating that equality of MRS's is a necessary condition for Pareto efficiency.

The first order conditions for the Lagrangian lead to
$\nu f_L = \epsilon = \mu g_L$
$\nu f_T = \eta = \mu g_T$.
Hence,
$f_L/f_T = \epsilon/\eta = g_L/g_T$.
$f_L/f_T = g_L/g_T = -\frac{\mathrm{dT}}{\mathrm{dL}}|_{f=constant} = -\frac{\mathrm{dT}}{\mathrm{dL}}|_{g=constant}$
, which represents the absolute value of the slope of the isoquant at the efficient points — the tangencies in the Edgeworth Box for inputs.

We want to show that Pareto efficiency requires that
$u^1_x/u^1_y = u^2_x/u^2_y = g_L/f_L = g_T/f_T$. The expression g_L/f_L (or g_T/f_T) is the marginal rate of transformation of x for y.

$g_L/f_L = \nu/\mu = g_T/f_T$.
But $u^1_x/u^1_y = \nu/\mu = u^2_x/u^2_y$
$u^1_x/u^1_y = \nu/\mu = u^2_x/u^2_y = g_L/f_L = \nu/\mu = g_T/f_T$.
Then we have $g_L/f_L = \frac{\partial \mathrm{y}}{\partial \mathrm{L^y}}/\frac{\partial \mathrm{x}}{\partial \mathrm{L^x}} = -\frac{\mathrm{dy}}{\mathrm{dx}}|_{L^x+L^y=L^o}$.
Similarly for g_T/f_T.

Equality of MRS_{xy}'s across individuals and equality of the MRS_{xy} to the MRT_{xy}.

$$\begin{aligned} MRT_{xy} = g_L/f_L = \frac{\partial \mathrm{y}}{\partial \mathrm{L^y}}/\frac{\partial \mathrm{x}}{\partial \mathrm{L^x}} &= -\frac{\mathrm{dy}}{\mathrm{dx}}|_{L^x+L^y=L^o} \\ = u^1_x/u^1_y = MRS^1_{xy} &= u^2_x/u^2_y = MRS^2_{xy} \end{aligned}$$

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$$\begin{aligned} MRT_{xy} = g_T/f_T = \frac{\partial \mathrm{y}}{\partial \mathrm{T^y}}/\frac{\partial \mathrm{x}}{\partial \mathrm{T^x}} &= -\frac{\mathrm{dy}}{\mathrm{dx}}|_{T^x+T^y=T^o} \\ = u^1_x/u^1_y = MRS^1_{xy} &= u^2_x/u^2_y = MRS^2_{xy} \end{aligned}$$

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First Fundamental Theorem of Welfare Economics: Competitive Equilibrium is Pareto Efficient

Let's describe $p^x, p^y, w, r, L^x, L^y, T^x, T^y, x^1, y^1, x^2, y^2$, that would constitute a general competitive equilibrium for the $2 \times 2 \times 2$ economy. The first order conditions are

$\frac{[\partial u^1/\partial x^1]}{[\partial u^1/\partial y^1]} = \frac{p^x}{p^y} = \frac{[\partial u^2/\partial x^2]}{[\partial u^2/\partial y^2]}$, that is marginal rates of substitution are equated to price ratios, and

$$p^x f_L = w = p^y g_L,$$
$$p^x f_T = r = p^y g_T \ .$$

That is, the marginal value product of factor inputs equals the factor prices. Competitive equilibrium is characterized by

$p^x/p^y = u^1_x/u^1_y$, by utility maximization ;
$p^x/p^y = u^2_x/u^2_y$, by utility maximization .
$p^x = w/f_L = r/f_T$, by profit maximization ;
$p^y = w/g_L = r/g_T$ by profit maximization.

But then it follows that $u^1_x/u^1_y = u^2_x/u^2_y = g_L/f_L = g_T/f_T$. And it follows that $g_L/g_T = w/r$, and $f_L/f_T = w/r$. But then it follows that $g_L/g_T = f_L/f_T$.

Thus $u^1_x/u^1_y = u^2_x/u^2_y = g_L/f_L = g_T/f_T$ and $g_L/g_T = f_L/f_T$. That is, common marginal rates of substitution in consumption equated to the marginal rate of transformation at an allocation where output is technically efficient (where firms have common marginal rates of technical substitution between inputs). This is the First Fundamental Theorem of Welfare Economics; a competitive equilibrium allocation is Pareto efficient.

This is the calculus treatment of the classical welfare economics. It is simple, instructive, useful. But what happens when the marginals are not well defined? At a corner solution. When commodities are sufficiently precisely specified, most solutions are corner solutions. Most consumptions of most goods are zeroes. That's why we use a separating hyperplane in Starr, Chapter 19