(b) $C'(q) = \sum_{j=1}^{J} c'_j(q^*_j)\, dq^*_j/dq = \sum_{j=1}^{J} \lambda\, dq^*_j/dq = \lambda\, d(\sum_{j=1}^{J} q^*_j)/dq = \lambda\, dq/dq = \lambda.$

Therefore, $C'(q) = c'_j(q^*_j)$ for all j.

(c) Each firm j solves $\underset{q_j}{\text{Max}}\ pq_j - c_j(q_j)$.

The first-order condition (assuming interior solution) is $c'_j(q_j) = p$.

If $p = C'(q)$, then we have $c'_j(q_j) = C'(q)$ for every j. If $c'_j(\cdot)$ is strictly increasing for all j, we must have $q_j = q^*_j(q)$ - the solution to the central authority's program in part (a) for total output q. Therefore, $\sum_{j=1}^{J} q_j = \sum_{j=1}^{J} q^*_j(q) = q$. In other words, if the market price is $C'(q)$, then the industry produces q. Therefore, $C'(\cdot)$ is the inverse of the industry supply function.

10.C.4. (a) The central authority's problem can be written as

$$\underset{(x_1,\ldots,x_I)\geq 0}{\text{Max}} \quad \sum_{i=1}^{I} \phi_i(x_i) \qquad \text{s.t.} \quad \sum_{i=1}^{I} x_i \leq x.$$

Assuming interior solution, the first-order condition is

$\phi'_i(x^*_i) = \lambda > 0$ for all i.

(b) $\gamma'(x) = \sum_{i=1}^{I} \phi'_i(x^*_i)\, dx^*_i/dx = \sum_{i=1}^{I} \lambda\, dx^*_i/dx = \lambda\, d(\sum_{i=1}^{I} x^*_i)/dx = \lambda\, dx/dx = \lambda.$

Therefore, $\gamma'(x) = \phi'_i(x^*_i)$ for all i.

(c) Each consumer solves $\underset{x_i}{\text{Max}}\ \phi_i(x_i) - Px_i$.

The first-order condition (assuming interior solution) is $\phi'_i(x_i) = P$.

If $P = \gamma'(x)$, then we have $\phi'_i(x_i) = \gamma'(x)$ for every i. If $\phi'_i(\cdot)$ is strictly decreasing for all i, we must have $x_i = x^*_i(x)$ - the solution to the central authority's program in part (a) above for total consumption x. Therefore, $\sum_{i=1}^{I} x_i = \sum_{i=1}^{I} x^*_i(x) = x$. In other words, if the market price is $\gamma'(x)$, then the aggregate demand is x. Therefore, $\gamma'(\cdot)$ is the inverse of the aggregate demand function.

10.C.5. The system of equations (10.C.4)-(10.C.6) here takes the following form:

$$\phi_i'(x_i^*) = p^* + t, \qquad i = 1, \ldots, I,$$

$$c_j'(q_j^*) = p^*, \qquad j = 1, \ldots, J,$$

$$\sum_{i=1}^{I} x_i^* = \sum_{j=1}^{J} q_j^*.$$

These equations describe the equilibrium (x^*, q^*, p^*) as an implicit function of t. Differentiating with respect to t, we get

$$\phi_i''(x_i^*)\, x_i^{*\prime}(t) = p^{*\prime}(t) + 1, \qquad i = 1, \ldots, I,$$

$$c_j''(q_j^*)\, q_j^{*\prime}(t) = p^{*\prime}(t), \qquad j = 1, \ldots, J,$$

$$\sum_{i=1}^{I} x_i^{*\prime}(t) = \sum_{j=1}^{J} q_j^{*\prime}(t).$$

This system of linear equations should be solved for $(x_i^{*\prime}(t), q_j^{*\prime}(t), p^{*\prime}(t))$. This can be easily done, for example, by expressing dx_i^*/dt and dq_j^*/dt from the first two sets of equations and substituting into the third equation. We obtain

$$(p^{*\prime}(t) + 1) \sum_{i=1}^{I} [\phi_i''(x_i^*)]^{-1} = p^{*\prime}(t) \sum_{j=1}^{J} [c_j''(q_j^*)]^{-1}.$$

From here we can express $p^{*\prime}(t)$:

$$p^{*\prime}(t) = \frac{\sum_{i=1}^{I} [\phi_i''(x_i^*)]^{-1}}{\sum_{i=1}^{I} [\phi_i''(x_i^*)]^{-1} - \sum_{j=1}^{J} [c_j''(q_j^*)]^{-1}}.$$

Compare to the expression on page 324 of the textbook.

10.C.6. **(a)** If the specific tax t is levied on the consumer, then he pays $p+t$ for every unit of the good, and the demand at market price p becomes $x(p+t)$.