18

U-Shaped Cost Curves and Concentrated Preferences

18.1 U-Shaped Cost Curves and Concentrated Preferences

In intermediate microeconomic theory, a firm's cost function is often described as U-shaped. The notion is that firms producing at low volume have high marginal costs. The marginal costs decline as volume increases and then start to rise again. There is a region of declining marginal costs. But declining marginal costs are inconsistent with convexity of technology, and convex technology is one of the assumptions used to show the existence of general equilibrium in Chapters 7, 11 and 17. Can we reconcile the elementary U-shaped cost curve model with the existence of general equilibrium?

Convexity of preferences was one of the assumptions used to demonstrate continuity or convexity of demand behavior needed for the proofs of existence of general equilibrium in Chapters 7, 11 and 17. But surely there are instances where convexity does not hold. A household might be equally pleased with a blue suit and a grey suit but half a blue suit and half a grey one is not so satisfactory. A resident may be equally satisfied with an apartment in San Francisco or one in Boston; half time in each is less satisfactory. The household has concentrated preferences (or a preference for concentrating consumption). Can these preferences be reconciled with the existence of general economic equilibrium?

We'll argue in this chapter that the answer is 'yes'. Using the Shapley-Folkman theorem we'll establish the existence of approximate equilibrium in these settings. The approximation will depend on the dimension of the commodity space, N. Holding N fixed while the number of firms #F and households #H becomes large (as in a fully competitive model), will allow the approximate equilibrium to be arbitrarily close to a full equilibrium as a proportion of the size of the economy.

The strategy of proof is to consider a fictional mathematical construct of an economy where we replace the (possibly nonconvex) typical firm's production technology Y^j with it's convex hull, $con(Y^j)$. We replace the households', $h \in H$, nonconvex preference contour sets, $A^h(x)$, by their convex hulls, $con(A^h(x))$. This fictional construct will fulfill the model of Chapter 17. It will have a market-clearing general equilibrium price vector p*. The artificial convex-valued supply and demand correspondences are formed from the convex hulls of the true underlying non-convex-valued supply and demand correspondences. Then the Shapley-Folkman Theorem implies that the market-clearing plans of the fictional convex-valued supply and demand correspondences are within a small bounded distance of the the true economy's underlying nonconvex-valued supply and demand correspondences. That is, the non-convex-valued demand and supply correspondences at p^* are nearly market-clearing. Further, the bound depends on the size of non-convexities in the original economy's sets, L, and on the dimension of the space, N, not on the number of firms or households in the economy. Thus, in a large economy, where the number of households in H becomes large, the average disequilibrium per household becomes small. Thus, in the limit as the economy becomes large (the setting where we expect the economy to behave competitively), the approximation to market clearing can be as close as you wish.

18.2 The Non-Convex Economy

We start with a model of the economy with the same notation and same assumptions as in Chapter 17 with the omission of two assumptions: P.I and C.VI(WC). Neither technology nor preferences are assumed to be convex.

18.2.1 Non-Convex Technology and Supply

Supply behavior of firms, $S^j(p)$, when it is well defined, may no longer be convex-valued. Since Y^j admits scale economies $S^j(p)$ may include many distinct points and not the line segments connecting them. A supply curve might look like figure 17.2. Alternatively, $S^j(p)$ might include 0 and a high level of output, but none of the values in between. This is, of course, the U-shaped cost curve case.

18.2.2 Non-Convex Preferences and Demand

Demand behavior of households, $D^i(p)$, when it is well defined, may no longer be convex-valued. Thus it is possible that $x, y \in D^i(p)$ but that

$\alpha x + (1-\alpha)y \notin D^i(p)$ for $0 < \alpha < 1$.

18.2.3 Nonexistence of Market Equilibrium

The proof of Theorem 17.7, relying on the Kakutani Fixed-Point Theorem, requires convexity of $S^j(p)$ for all $j \in F$ and of $D^i(p)$ for all $i \in H$. Theorem 17.7 cannot be applied to the non-convex economy. We cannot rely on the existence of general competitive equilibrium. What can go wrong? Roughly, a demand curve (or correspondence) can run through the holes in a supply curve (or correspondence), resulting in no nonull intersection and no equilibrium prices.

18.3 Artificial Convex Counterpart to the Non-Convex Economy

We now form a convex counterpart to the non-convex economy. This artificial convex economy will be designed to fulfill the conditions of Chapter 17 and sustain competitive general equilibrium prices. We will then show, using the Shapley-Folkman Theorem, that the equilibrium price vector of the artificial convex economy supports an approximate equilibrium allocation of the original non-convex economy. The remaining disequilibrium (unsatisfied demand and supply at these prices) is independent of the size of the economy, as measured by the number of households, total output, or number of firms. Hence as a proportion of a large economy the remaining disequilibrium can be arbitrarily small.

18.3.1 Convexified Technology and Supply

Starting from the non-convex technology set Y^j, we merely substitute its convex hull, $con(Y^j)$, for each $j \in F$. Then substitute the convex hull of the aggregate technology set for the aggregate set Y, $con(Y) = con(\sum_{j\in F} Y^j) = \sum_{j\in F} con(Y^j)$. Then we assume the convexified counterpart to P.III (the notation K is intended as a nmemonic for "convex")

PK.III $con(Y^j)$ is closed for all $j \in F$.

The economic implication of PK.III is that scale economies are bounded — as in the U-shaped cost curve case; average costs are not indefinitely diminishing. Thus, for example,

$Y^j = \{(x, y) | y \leq (-x)^2, x \leq 0\}$ would not fulfill PK.III but

$Y'^j = \{(x, y) | y \leq (-x)^2, \text{ for } -1 \leq x \leq 0, y \leq \sqrt{-x} \text{ for } x \leq -1\}$ would fulfill PK.III.

Now we introduce a counterpart to P.IV for the convexified economy.

PK.IV (a) if $y \in con(Y)$ and $y \neq 0$, then $y_k < 0$ for some k.

(b) if $y \in con(Y)$ and $y \neq 0$, then $-y \notin con(Y)$.

Then we consider a production sector characterized by firms with technologies $con(Y^j)$ for all $j \in F$. We assume P.II, PK.III, PK.IV. Since the technology of each firm j is $con(Y^j)$, P.I is trivially fulfilled. Then the production sector fulfills all of the assumptions of Theorem 17.7.

The artificially convex supply behavior of firm k then is

$S^{kj}(p) \equiv \{y^o \in con(Y^j) | p \cdot y^o \geq p \cdot y \text{ for all } y \in con(Y^j)\}$.

The artificially convex profit function of firm j is

$\pi^{kj}(p) \equiv p \cdot y^o$, where $y^o \in S^{kj}(p)$.

Under PK.III, a typical point of $S^{kj}(p)$ will be a point of $S^j(p)$ or a convex combination of points of $S^j(p)$.

Lemma: Assume P.II, PK.III, PK.IV and suppose $S^{kj}(p)$ is nonempty (exists and is well defined). Then $y^j \in S^{kj}(p)$ implies $y^j \in con(S^j(p))$ and $\pi^{kj}(p) = \pi^j(p)$.

18.3.2 Artificial Convex Preferences and Demand

Household h's budget set $B^h(p)$ is described in Chapter 17, and as in Chapter 17, there may be price vectors where $B^h(p)$ is not well defined.

The formal definition of h's demand behavior $D^h(p)$ is precisely the same as in Chapter 17. However, without the convexity assumption, C.VI(WC), on $\succeq_i$ the demand correspondence $D^h(p)$ may look rather different. $D^h(p)$ will be upper hemicontinuous in neighborhoods where it is well defined, but it may include gaps that look like jumps in demand behavior. That's because $D^h(p)$ may not be convex-valued.

In order to pursue the plan of the proof we need to formalize the notion of artificially convex preferences.

Definition: Let $x, y \in X^i$. We say $x \succeq_{ki} y$ if for every $w \in X^i$, $y \in con(A^i(w))$ implies $x \in con(A^i(w))$.

This definition creates a convex preference ordering $\succeq_{ki}$ for household i, by substituting the family of convex hulls of i's upper contour sets $con(A^i(w))$ for i's original upper contour sets $A^i(w)$. Without going more deeply into the geometry of these new upper contour sets, it is sufficient to assume

(CK.IV) For each $i \in H$, $\succeq_{ki}$ fulfills C.IV.

(CK.V) For each $i \in H$, $\succeq_{ki}$ fulfills C.V.

(CK.VI) For each $i \in H$, $\succeq_{ki}$ fulfills C.VI(WC).

We need to develop the notion of a convex-valued counterpart to $D^h(p)$. Define $D^{kh}(p) \equiv \{x^o | x^o \in B^h(p), x^o \succeq_{kh} x \text{ for all } x \in B^h(p)\}$. Under assumptions CK.IV, CK.V, CK.VI, $D^{kh}(p)$ is very well behaved in neighborhoods where it is well defined: upper hemicontinuous, convex-valued. Using $\succeq_{kh}$ as the preference ordering, rather than the nonconvex ordering $\succeq_h$, fills in the gaps left in $D^h(p)$ by the nonconvex ordering. The typical point in $D^{kh}(p)$ will either be a point of $D^h(p)$ or the convex combination of points of $D^h(p)$.

Lemma: Assume C.I, C.II, C.III,CK.IV, CK.V, CK.VI, C.VII. Let $p \in P$, and suppose $M^i(p)$ exists. Then $x^i \in D^{ki}(p)$ implies $x^i \in con(D^i(p))$.

18.3.3 Competitive Equilibrium in the Artificial Convex Economy

One of the great powers of mathematics is that you only have to solve a problem once: when it reappears, you already know the answer. Even when it reappears under a new wrapping, if it's the same underneath you can say "reduced to the previous case." That's what we've been working on in sections 1.3.1 and 1.3.2: taking the non-convex economy of section 1.2 and restating it in a fashion where we can reduce consideration of its general equilibrium to a "previous case," the model of Chapter 17.

Consider a convex economy characterized in the following way:

Firms: $j \in F$, technologies are $con(Y^j)$, fulfilling P.I, P.II, PK.III, PK.IV.

Households: $i \in H$, tastes $\succeq_{ki}$, fulfilling C.I, C.II, C.III,CK.IV, CK.V, CK.VI, C.VII; endowments r^i, firm shares α^{ij}.

Then this economy fulfills all of the assumptions of Theorem 17.7. Applying that theorem, we know the convex economy has a general competitive equilibrium. That is,

Lemma: Assume P.II, PK.III, PK.IV, C.I, C.II, C.III,CK.IV, CK.V, CK.VI, C.VII. Then there are prices $p^o \in P$, production plans $y^{oj} \in S^{kj}(p^o)$, consumption plans $x^{oi} \in D^{ki}(p^o)$ so that markets clear

$$\sum_{i \in H} r^i + \sum_{j \in F} y^{oj} \geq \sum_{i \in H} x^{oi}$$

where the inequality applies co-ordinatewise, and $p^o_n = 0$ for n so that the strict inequality holds.

Of course, the result of this lemma, in itself, should be of no interest at all. After all, the convex economy, is a figment of our imagination. The real economy is non-convex. But now we can apply the power of mathematics. The Shapley-Folkman Theorem (Chapter 2, section 2.9) tells us that the actual economy is very near the artificial convex economy described above.

This leads us to the result in the next section: the equilibrium of the constructed convex economy above is very nearly an equilibrium of the original non-convex economy.

18.4 Approximate Equilibrium

We now use the artificial convex economy set up above and the Corollary to the Shapley-Folkman Theorem to establish the existence of an approximate equilibrium in an economy with bounded non-convexities.

Recall

Definition: We define the inner radius of $S \subset R^N$ as
$r(S) \equiv \sup_{x \in con(S)} \inf_{T \subset S; x \in con(T)} rad(T)$

Corollary to the Shapley-Folkman Theorem: Let F be a finite family of compact subsets $S \subset R^N$ and $L > 0$ so that $r(S) \leq L$ for all $S \in F$. Then for any $x \in con(\sum_{S \in F} S)$ there is $y \in \sum_{S \in F} S$ so that $|x - y| \leq L\sqrt{N}$.

Now we can apply this corollary to establish the existence of an approximate equilibrium.

Theorem 18.1 Let the economy fulfill P.II, PK.III, PK.IV. and C.I, C.II, C.III, CK.IV, CK.V, CK.VI, C.VII. Let there be $L > 0$ so that for all $h \in H$, $x \in X^h$, $f \in F$,

$$r(A^h(x)) \leq L, \text{ and } r(Y^f) \leq L.$$

Then there are prices $p^* \in P$, production plans $y^{\dagger j} \in Y^j$, $y^{*j} \in con(Y^j)$, consumption plans $x^{*i} \in X^i$, and $x^{\dagger i} \in X^i$ so that

$$\sum_{i \in H} x^{*i} \leq \sum_{j \in F} y^{*j} + r$$

$$p_k^* = 0 \text{ for } k \text{ so that } \sum_{i \in H} x_k^{*i} < \sum_{j \in F} y_k^{*j} + r_k$$

$$p^* \cdot x^{\dagger i} = p^* \cdot r^i + \sum_{j \in F} \alpha^{ij} p^* \cdot y^{\dagger j} = p^* \cdot r^i + \sum_{j \in F} \alpha^{ij} p^* \cdot y^{*j}$$

$$x^{\dagger i} \text{ maximizes } u^i(x) \text{ subject to } p^* \cdot x \leq p^* \cdot r^i + \sum_{j \in F} \alpha^{ij} p^* \cdot y^{\dagger j}$$

$$|[\sum_{i\in H} x^{*i} - \sum_{j\in F} y^{*j}] - [\sum_{i\in H} x^{\dagger i} - \sum_{j\in F} y^{\dagger j}]| \leq L\sqrt{N}$$

Proof By the lemma of section 18.3.3 there is $p^* \in P$, $y^{*j} \in S^{kj}(p^*)$, $x^{*i} \in D^{ki}(p^*)$ so that

$$\sum_{i\in H} r^i + \sum_{j\in F} y^{*j} \geq \sum_{i\in H} x^{*i} \text{ , with } p^*_k = 0 \text{ for } k \text{ so that a strict inequality holds,}$$

and $p^* \cdot x^{*i} = p^* \cdot r^i + \sum_{j\in F} \alpha^{ij} p^* \cdot y^{*j}$.

Using the lemmata of 18.3.1 and 18.3.2, $y^{*j} \in con(S^j(p^*))$ and $x^{*i} \in con(D^i(p^*))$. Applying the Corollary to the Shapley-Folkman Theorem, for each $j \in F$ there is $y^{\dagger j} \in S^j(p^*)$, and for each $i \in H$ there is $x^{\dagger i} \in D^i(p^*)$ so that

$$|[\sum_{i\in H} x^{*i} - \sum_{j\in F} y^{*j}] - [\sum_{i\in H} x^{\dagger i} - \sum_{j\in F} y^{\dagger j}]| \leq L\sqrt{N}$$

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QED

The theorem says that there are prices p^* so that households and firms can choose plans that are optimizing at p^* , fulfilling budget constraint, with the allocations nearly (but not perfectly) market clearing. The proof is a direct application of the Corollary to the Shapley-Folkman Theorem and the Lemma of section 18.3.3, above. The Lemma establishes the existence of market clearing prices for an 'economy' characterized by the convex hulls of the actual economy. Then applying the Corollary to the Shapley-Folkman Theorem there is a choice of approximating elements in the original economy that is within the bound $L\sqrt{N}$ of the equilibrium allocation of the artificial convex economy.

18.5 Bibliographic Note

The treatment here parallels Kenneth Arrow and Frank Hahn, General Competitive Analysis chapter 7. The demonstration of an approximate equilibrium in a pure exchange economy using the Shapley-Folkman Theorem appears originally in R. Starr, "Quasi-Equilibria in Markets with Nonconvex Preferences," Econometrica, 1969. The limiting case with a continuum (uncountable infinity) of households is developed in R. J. Aumann, "Existence of Competitive Equilibrium in Markets with a Continuum of Traders," Econometrica, 1966, and in W. Hildenbrand, Core and Equilibria of a Large Economy, 1974.

Exercises