Economics 200B Prof. R. Starr UCSD Winter 2009

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Bargaining and equilibrium: The core of a market economy

Set $X^i = \mathbf{R}^N_+$, all i.

Each $i \in H$ has an endowment $r^i \in \mathbf{R}^N_+$ and a preference quasi-ordering $\succeq_h$ defined on $\mathbf{R}^N_+$.

An allocation is an assignment of $x^i \in \mathbf{R}^N_+$ for each $i \in H$. A typical allocation, $x^i \in \mathbf{R}^N_+$ for each $i \in H$, will be denoted $\{x^i, i \in H\}$. An allocation, $\{x^i, i \in H\}$, is feasible if $\sum_{i \in H} x^i \leq \sum_{i \in H} r^i$, where the inequality holds coordinatewise.

We assume preferences fulfill weak monotonicity (C.IV**), continuity (C.V), and strict convexity (C.VI(SC)).

The core of a pure exchange economy

Definition A ***coalition*** is any subset $S \subseteq H$. Note that every individual comprises a (singleton) coalition.

Definition An allocation $\{x^i, h \in H\}$ is ***blocked*** by $S \subseteq H$ if there is a coalition $S \subseteq H$ and an assignment $\{y^i, i \in S\}$ so that:

(i) $\sum_{i \in S} y^i \leq \sum_{i \in S} r^i$ (where the inequality holds coordinatewise),
(ii) $y^i \succeq_i x^i$, for all $i \in S$, and
(iii) $y^h \succ_h x^h$, for some $h \in S$

Definition The ***core*** of the economy is the set of feasible allocations that are not blocked by any coalition $S \subseteq H$.

- Any allocation in the core must be individually rational. That is, if $\{x^i, i \in H\}$ is a core allocation then we must have $x^i \succeq_h r^i$, for all $i \in H$.
- Any allocation in the core must be Pareto efficient.

(i) The competitive equilibrium is always in the core (Theorem 13.1).

Theorems 14.2 and 14.3 say that

(ii) For a large economy, the set of competitive equilibria and the core are virtually identical. All core allocations are (nearly) competitive equilibria.

The competitive equilibrium allocation is in the core

Definition $p \in \mathbf{R}_+^N$, $p \neq 0$, $x^i \in \mathbf{R}_+^N$, for each $i \in H$, constitutes a competitive equilibrium if

(i) $p \cdot x^i \leq p \cdot r^i$, for each $i \in H$,
(ii) $x^i \succeq_i y$, for all $y \in R_+^N$, such that $p \cdot y \leq p \cdot r^i$, and
(iii) $\sum_{i \in H} x^i \leq \sum_{i \in H} r^i$ (the inequality holds coordinatewise) with $p_k = 0$ for any $k = 1, 2, \ldots, N$ so that the strict inequality holds.

Theorem 13.1 Let the economy fulfill C.II, C.IV**, C.VI(SC) and let $X^i = \mathbf{R}_+^N$. Let p, x^i, $i \in H$, be a competitive equilibrium. Then $\{x^i, i \in H\}$ is in the core of the economy.

Proof We will present a proof by contradiction. Suppose the theorem were false. Then there would be a blocking coalition $S \subseteq H$ and a blocking assignment $y^i, i \in S$. We have

$\sum_{i \in S} y^i \leq \sum_{i \in S} r^i$ (attainability, the inequality holds coordinatewise)
$y^i \succeq_i x^i$, for all $i \in S$, *and*
$y^h \succ_h x^h$, some $h \in S$.

But x^i is a competitive equilibrium allocation. That is, for all $i \in H$, $p \cdot x^i = p \cdot r^i$ (recalling Lemma 10.1), and $x^i \succeq_i y$, for all $y \in R_+^N$ such that $p \cdot y \leq p \cdot r^i$.

Note that $\sum_{i \in S} p \cdot x^i = \sum_{i \in S} p \cdot r^i$. Then for all $i \in S$, $p \cdot y^i \geq p \cdot r^i$. That is, x^i represents i's most desirable consumption subject to budget constraint. y^i is at least as good under preferences $\succeq_i$ fulfilling C.II, C.IV, C.VI(SC), (local non-satiation). Therefore, y^i must be at least as expensive. Furthermore, for h, we must have $p \cdot y^h > p \cdot r^h$. Therefore, we have

$$\sum_{i \in S} p \cdot y^i > \sum_{i \in S} p \cdot r^i.$$

Note that this is a strict inequality. However, for coalitional feasibility we must have

$$\sum_{i \in S} y^i \leq \sum_{i \in S} r^i.$$

But since $p \geq 0$, $p \neq 0$, we have $\sum_{i \in S} p \cdot y^i \leq \sum_{i \in S} p \cdot r^i$. This is a contradiction. The allocation $\{y^i, i \in S\}$ cannot simultaneously be smaller or equal to the sum of endowments r^i coordinatewise and be more expensive at prices $p, p \geq 0$. The contradiction proves the theorem. QED

Convergence of the core of a large economy

Replication; a large economy

In replication, the economy keeps cloning itself.

duplicate to triplicate, ..., to Q-tuplicate, and so on, the set of core allocations keeps getting smaller, although it always includes the set of competitive equilibria (per Theorem 13.1).

Q-fold replica economy, denoted Q-H. $Q = 1, 2, \ldots$.

$\#H \times Q$ agents.

Q agents with preferences $\succeq_1$ and endowment r^1,

Q agents with preferences $\succeq_2$ and endowment $r^2, \ldots,$ and Q agents with preferences $\succeq_{\#H}$ and endowment $r^{\#H}$. Each household $i \in H$ now corresponds to a household type. There are Q individual households of type i in the replica economy Q-H.

Competitive equilibrium prices in the original H economy will be equilibrium prices of the Q-H economy. Household i's competitive equilibrium allocation x^i in the original H economy will be a competitive equilibrium allocation to all type i households in the Q-H replica economy. Agents in the Q-H replica economy will be denoted by their type and a serial number. Thus, the agent denoted i, q will be the qth agent of type i, for each $i \in H, q = 1, 2, \ldots, Q$.

Equal treatment

Theorem 14.1 (Equal treatment in the core) Assume C.IV, C.V, and C.VI(SC). Let $\{x^{i,q}, i \in H, q = 1, \ldots, Q\}$ be in the core of Q-H, the Q-fold replica of economy H. Then for each $i, x^{i,q}$ is the same for all q. That is, $x^{i,q} = x^{i,q'}$ for each $i \in H, q \neq q'$.

Proof of Theorem 14.1 Recall that the core allocation must be feasible. That is,

$$\sum_{i \in H} \sum_{q=1}^{Q} x^{i,q} \leq \sum_{i \in H} \sum_{q=1}^{Q} r^i.$$

Equivalently,

$$\frac{1}{Q} \sum_{i \in H} \sum_{q=1}^{Q} x^{i,q} \leq \sum_{i \in H} r^i.$$

Suppose the theorem to be false. Consider a type i so that $x^{i,q} \neq x^{i,q'}$. For each type i, we can rank the consumptions attributed to type i according to $\succeq_i$.

For each i, let x^{i^*} denote the least preferred of the core allocations to type $i, x^{i,q}, q = 1, \ldots, Q$. For some types i, all individuals of the type will have the same consumption and x^{i^*} will be this expression. For those in which the consumption differs, x^{i^*} will be the least desirable of the consumptions of the type. We now form a coalition consisting of one member of each type: the individual from each type carrying the worst core allocation, x^{i^*}.

Consider the average core allocation to type i, to be denoted $\bar{x}^i$.

$\bar{x}^i = \frac{1}{Q}\sum_{q=1}^{Q} x^{i,q}$.

We have, by strict convexity of preferences (C.VI(SC)),

$$\bar{x}^i = \frac{1}{Q}\sum_{q=1}^{Q} x^{i,q} \succ_i x^{i^*} \text{ for those types } i \text{ so that } x^{i,q} \text{ are not identical,}$$

and

$$x^{i,q} = \bar{x}^i = \frac{1}{Q}\sum_{q=1}^{Q} x^{i,q} \sim_i x^{i^*} \text{ for those types } i \text{ so that } x^{i,q} \text{ are identical.}$$

From feasibility, above, we have that

$$\sum_{i\in H} \bar{x}^i = \sum_{i\in H} \frac{1}{Q}\sum_{q=1}^{Q} x^{i,q} = \frac{1}{Q}\sum_{i\in H}\sum_{q=1}^{Q} x^{i,q} \leq \sum_{i\in H} r^i.$$

In other words, a coalition composed of one of each type (the worst off of each) can achieve the allocation $\bar{x}^i$. However, for each agent in the coalition, $\bar{x}^i \succeq_i x^{i^*}$ for all i and $\bar{x}^i \succ_i x^{i^*}$ for some i. Therefore, the coalition of the worst off individual of each type blocks the allocation $x^{i,q}$. The contradiction proves the theorem. QED

Core$(Q) = \{x^i, i \in H\}$ where $x^{i,q} = x^i, q = 1, 2, \ldots, Q$, and the allocation $x^{i,q}$ is unblocked.

<u>Core convergence in a large economy</u>

As Q grows there are more blocking coalitions, and they are more varied. Any coalition that blocks an allocation in Q-H still blocks the allocation in $(Q+1)$-H, but there are new blocking coalitions and allocations newly blocked in $(Q+1)$-H.

Recall the Bounding Hyperplane Theorem:

Theorem 2.11, Bounding Hyperplane Theorem (Minkowski) Let K be convex, $K \subseteq \mathbf{R}^N$. There is a hyperplane H through z and bounding for K if z is not interior to K. That is, there is $p \in \mathbf{R}^N, p \neq 0$, so that for each $x \in K, p \cdot x \geq p \cdot z$.

Theorem 14.2 (Debreu-Scarf) Assume C.IV**, C.V, C.VI(SC), and let $X^i = \mathbf{R}^N_+$. Let $\{x^{\circ i}, i \in H\} \in \text{core}(Q)$ for all $Q = 1, 2, 3, 4, \dots$. Then $\{x^{\circ i}, i \in H\}$ is a competitive equilibrium allocation for Q-H, for all Q.

Proof We must show that there is a price vector p so that for each household type $i, p \cdot x^{\circ i} \leq p \cdot r^i$ and that $x^{\circ i}$ optimizes preferences $\succeq_i$ subject to this budget.

For each $i \in H$, let $\Gamma^i = \{z \mid z \in \mathbf{R}^N, z + r^i \succ_i x^{oi}\}$.

Let $\Gamma = \{\sum_{i \in H} a_i z^i \mid z^i \in \Gamma^i,\ a_i \geq 0, \sum a_i = 1\}$, the set of convex combinations of preferred net trades.

Γ is the convex hull of the union of the sets Γ^i.

Note that $(x^{\circ i} - r^i) \in \text{boundary}(\Gamma^i)$,

$(x^{\circ i} - r^i) \in \overline{\Gamma}^i$, and

$(x^{\circ i} - r^i) \in \text{boundary}(\Gamma)$ for all i.

Claim: $0 \notin int(\Gamma)$. We will show that the possibility that $0 \in int(\Gamma)$ corresponds to the possibility of forming a blocking coalition against the core allocation x^{oi}, a contradiction. Suppose that $0 \in int(\Gamma)$.

If $0 \in int(\Gamma)$, then there is an ε-neighborhood about 0 ($\varepsilon > 0$) contained in $int(\Gamma)$ (Recall that $X^i \equiv \mathbf{R}^N_+$) . The typical element of $int(\Gamma)$ can be represented as $\sum a_i z^i$, where $z^i \in \Gamma^i$.

Let $\mathbf{R}^N_-$ denote the nonpositive quadrant of $\mathbf{R}^N$. Take the intersection $int(\Gamma) \cap \mathbf{R}^N_-$, that is, the nonpositive quadrant of $int(\Gamma)$. Choose $z \in int(\Gamma) \cap \mathbf{R}^N_-$ so that $z = \sum a_i z^i$ with a_i rational for all i. This is possible since $\varepsilon > 0$ and any real a_i can be approximated arbitrarily closely by a sequence of rationals.

Find a common denominator for the a_i. Consider Q equal to the common denominator of the a_i (this is how replication with large Q overcomes the indivisibility of the individual agents). We have $\sum a_i z^i \leq 0$ (coordinatewise). We wish to show that this implies the presence of a blocking coalition against the allocation x^{oi} in H-Q, where Q is the common denominator of the a_i. Form the coalition S, consisting of Qa_i (an integer) of type i agents, $i \in H$. Consider the allocation $x'^i = r^i + z^i$ to agents in S. Note that $x'^i \succ_i x^{oi}$ (by definition of Γ^i). We have $\sum a_i z^i \leq 0$. Thus $\sum (Qa_i) z^i \leq 0$.

But then we have $\sum (Qa_i)(x'^i - r^i) \leq 0$ or, equivalently, $\sum (Qa_i) x'^i \leq$

$\sum(Qa_i)r^i$, which means x'^i is attainable by S. But x'^i improves upon x^{oi} according to the preferences of $i \in S$. Thus S blocks x^{oi}, which is a contradiction. Hence, as claimed, $0 \notin int(\Gamma)$.

$0 \in$ boundary of Γ. This occurs inasmuch as $0 = (1/\#H)\sum_{i\in H}(x^{\circ i} - r^i)$, and the right-hand side of this expression is an element of $\overline{\Gamma}$, the closure of Γ. Thus 0 represents just the sort of boundary point through which a supporting hyperplane may go in the Bounding Hyperplane Theorem. The set Γ is trivially convex.

Bounding Hyperplane Theorem. There is $p \in \mathbf{R}^N, p \neq 0$, so that for all $v \in \Gamma$, $p \cdot v \geq p \cdot 0 = 0$. Noting $X^i = \mathbf{R}^N_+$, C.IV** (implying local non-satiation) , we know that $p \geq 0$. Now $(x^{\circ i} - r^i) \in \overline{\Gamma}$ for each i, so $p \cdot (x^{\circ i} - r^i) \geq 0$. But $\sum_{i\in H}(x^{\circ i} - r^i) = 0$, so $p \cdot \sum_{i\in H}(x^{\circ i} - r^i) = 0$. Hence $p \cdot (x^{\circ i} - r^i) = 0$ each i. Equivalently, $p \cdot x^{\circ i} = p \cdot r^i$. This gives us

$$0 = p \cdot \sum_{i\in H} \frac{1}{\#H}(x^{\circ i} - r^i) = \inf_{x\in\Gamma} p \cdot x = \sum_{i\in H} \frac{1}{\#H}\left[\inf_{z^i\in\Gamma^i} p \cdot z^i\right],$$

so

$$p \cdot (x^{\circ i} - r^i) = \inf_{z^i\in\Gamma^i} p \cdot z^i.$$

We have then for each i, that $p \cdot (x^{\circ i} - r^i) = \inf p \cdot y$ for $y \in \Gamma^i$. Equivalently, $x^{\circ i}$ minimizes $p \cdot (x - r^i)$ subject to $x \succeq_i x^{\circ i}$. In addition, $p \cdot x^{\circ i} = p \cdot r^i$. Further,there is an ε-neighborhood of $x^{\circ i}$ contained in X^i. By C.IV**, C.V, expenditure minimization subject to a utility constraint is equivalent to utility maximization subject to budget constraint. Hence $x^{\circ i}, i \in H$, is a competitive equilibrium allocation. QED

A Large Economy without Replication

The Shapley-Folkman Lemma

The convex hull of a set S will be the smallest convex set containing S. The convex hull of S will be denoted con(S). We can define con(S) , for $S \subset R^N$ as follows

$$\text{con(S)} \equiv \{x \mid x = \sum_{i=0}^{N} \alpha^i x^i, \text{where } x^i \in \text{S}, \alpha^i \geq 0 \text{ all i, and } \sum_{i=0}^{N} \alpha^i = 1\}.$$

or equivalently as

$$\text{con(S)} \equiv \bigcap_{S\subset\ T;T\ convex} T.$$

That is con(S) is the smallest convex set in R^N containing S.

Lemma (Shapley-Folkman): Let S^1, S^2, S^3, …,S^m, be nonempty compact subsets of R^N. Let $x \in con(S^1 + S^2 + S^3 + \ldots + S^m)$. Then for each i=1,2,…,m, there is $y^i \in con(S^i)$ so that $\sum_{i=1}^m y^i = x$ and with at most N exceptions, $y^i \in S^i$. Equivalently: Let F be a finite family of nonempty compact sets in R^N and let $y \in con(\sum_{S\in F} S)$. Then there is a partition of F into two disjoint subfamilies F' and F'' with the number of elements in $F' \leq N$ so that $y \in \sum_{S\in F'} con(S) + \sum_{S\in F''} S$.

We start by measuring the largest of the individual endowments. Define

$$M \equiv \max\{\sum_{i\in S} r^i_n | n = 1, ..., N, S \subseteq H, \#S = N\}$$

Theorem 14.3: Assume C.IV**, $X^i = \mathbf{R}^N_+$, for all $i \in H$, a pure exchange economy. Let $\{x^{\circ i} | i \in H\}$ be a core allocation for H. Then there is $p \in P$ so that

(i) $\sum_{i\in H} |p \cdot (x^{\circ i} - r^i)| \leq 2M$

(ii) $\sum_{i\in H} |\inf\{p \cdot (x - r^i) | x \succ_i x^{\circ i}\}| \leq 2M$

Proof: Define Γ^i as in the proof of Theorem 14.2. $\Gamma^i = \{z \mid z \in \mathbf{R}^N, z + r^i \succ_i x^{oi}\}$. Define $\Omega \equiv \sum_{i\in H}\{\Gamma^i \cup \{0\}\}$.

The proof proceeds in several steps.

Step 1: Let R^N_{++} denote the strictly positive quadrant of R^N, that is, the interior of R^N_+. We claim $(-R^N_{++}) \cap \Omega = \emptyset$. The reason is straightforward. If there is a nonempty intersection we can form a blocking coalition and block the core allocation — but of course, the core is unblocked, so this leads to a contradiction.

Suppose contrary to the claim there is $z \in \Omega$ so that $z << 0$. Then there is $z^i \in \{\Gamma^i \cup \{0\}\}$ for each $i \in H$ so that $\sum_{i\in H} z^i << 0$. Take the subset $S \subset H$ of $i \in H$ corresponding to the nonzero elements z^i in this sum. Then for $i \in S$ there is $z^i \in \Gamma^i$ so that $\sum_{i\in S} z^i < 0$ (the inequality holds co-ordinatewise). But then S is a blocking coalition. That is for all $i \in S$, $z^i = x'^i - r^i$ so that $x'^i \succ_i x^{oi}$ and $\sum_{i\in S} x'^i \leq \sum_{i\in S} r^i$. This is a contradiction. Hence we have $(-R^N_{++}) \cap \Omega = \emptyset$ as claimed.

Step 2: Recall that the notation $con(A)$ denotes the convex hull of the set A. Define the set Z as the strictly negative quadrant of R^N translated to the southeast by M in each co-ordinate. That is, let

$Z \equiv \{z \in R^N | z_n < -M, \text{ for } n = 1, 2, ..., N\}$. In this step, we establish that $Z \cap con(\Omega) = \emptyset$.

Again, we use a proof by contradiction, establishing a blocking coalition in the event that the step were not fulfilled. Suppose contrary to the step, we have $Z \cap con(\Omega) \neq \emptyset$. Choose $z \in Z \cap con(\Omega)$. Then by the Shapley-Folkman Lemma we can represent z in the following way. There is a partition of H into disjoint subsets S and T with no more than N elements in T. There is a choice of $z^i \in con(\{\Gamma^i \cup \{0\}\})$ so that $z = \sum_{i \in S} z^i + \sum_{i \in T} z^i$, where for all $i \in S, z^i \in \{\Gamma^i \cup \{0\}\}$ and for all $i \in T, z^i \in [con(\{\Gamma^i \cup \{0\}\}) \backslash \{\Gamma^i \cup \{0\}\}]$. That is, a point in the convex hull of Ω is the sum of points of $con(\{\Gamma^i \cup \{0\}\})$ no more than N of which are from $[con(\{\Gamma^i \cup \{0\}\}) \backslash \{\Gamma^i \cup \{0\}\}]$. That is, most of the summands making up the convex hull of the sum will be from the original sets of the sum while a fixed finite number will be from the corresponding convex hulls. The original sum was nearly convex on its own.

Recall that for each i, $0 \in \{\Gamma^i \cup \{0\}\}$ and that $z << -(M, M, ..., M)$. Then the sum

$[\sum_{i \in S} z^i + \sum_{i \in T} 0] \in \Omega$. Note that each element of $con(\Gamma^i \cup \{0\}) \geq -r^i$ (the inequality applies co-ordinatewise). Then we have $[\sum_{i \in S} z^i + \sum_{i \in T} 0] = z - \sum_{i \in T} z^i \leq z + \sum_{i \in T} r^i << -(M, M, ..., M) + \sum_{i \in T} r^i \leq 0$. But then $(-R^N_{++}) \cap \Omega \neq \emptyset$ contradicting Step 1. The contradiction suffices to establish Step 2.

Step 3: By the Separating Hyperplane Theorem, there is $p^* \neq 0, p^* \geq 0$ (by C.IV**) and real k so that $p^* \cdot x \geq k \geq p^* \cdot y$ for all $x \in con(\Omega), y \in Z$. Then without loss of generality we take $p^* \in P$.

Step 4: $(x^{\circ i} - r^i) \in \bar{\Gamma}^i$ (the closure of Γ^i) so

$p^* \cdot (x^{\circ i} - r^i) \geq \inf_{i \in H} \{p^* \cdot y | y \in \Gamma^i \cup \{0\}\}$. Let H^+ denote the subset of H so that $p^* \cdot (x^{\circ i} - r^i) \geq 0$. Let H^- denote the subset of H so that $p^* \cdot (x^{\circ i} - r^i) < 0$.

It is useful here to establish an identity

$\sum_{i \in H^+} \inf\{p^* \cdot y | y \in \Gamma^i \cup \{0\}\} + \sum_{i \in H^-} \inf\{p^* \cdot y | y \in \Gamma^i \cup \{0\}\} = \inf\{p^* \cdot y | y \in \Omega\}$

$$\begin{aligned}
&\sum_{i \in H^+} p^* \cdot (x^{\circ i} - r^i) \geq \sum_{i \in H^+} \inf\{p^* \cdot y | y \in \Gamma^i \cup \{0\}\} \\
&\geq \sum_{i \in H^+} \inf\{p^* \cdot y | y \in \Gamma^i \cup \{0\}\} + \sum_{i \in H^-} \inf\{p^* \cdot y | y \in \Gamma^i \cup \{0\}\} \\
&= \sum_{i \in H} \inf\{p^* \cdot y | y \in \Gamma^i \cup \{0\}\} \ = \ \inf\{p^* \cdot y | y \in \Omega\} \\
&= \inf\{p^* \cdot y | y \in con(\Omega)\} \geq k \ \geq \sup\{p^* \cdot y | y \in Z\} = -M \ .
\end{aligned}$$

The core allocation $x^{\circ i}$ is attainable, so $\sum_{i \in H} (x^{\circ i} - r^i) \leq 0$ and for any goods n in surplus at the core allocation $p^*_n = 0$. So $\sum_{i \in H} p^* \cdot (x^{\circ i} - r^i) = 0$. Then $\sum_{i \in H^-} p^* \cdot (x^{\circ i} - r^i) = -\sum_{i \in H^+} p^* \cdot (x^{\circ i} - r^i) \geq \inf\{p^* \cdot y | y \in \Omega\} \geq -M$

This implies that

$$M \geq -\inf\{p^* \cdot y | y \in \Omega\} \geq \sum_{i \in H^+} p^* \cdot (x^{\circ i} - r^i) \quad (*)$$

Note that for $i \in H^+$, $\inf\{p^* \cdot y | y \in \Gamma^i \cup \{0\}\} \leq 0 \quad (**)$

Now the conclusions of the theorem follow directly.

$\sum_{i\in H^-} |p^* \cdot (x^{\circ i} - r^i)| = \sum_{i\in H^+} |p^* \cdot (x^{\circ i} - r^i)| \leq M$, so

$\sum_{i\in H} |p^* \cdot (x^{\circ i} - r^i)| = \sum_{i\in H^-} |p^* \cdot (x^{\circ i} - r^i)| + \sum_{i\in H^+} |p^* \cdot (x^{\circ i} - r^i)| \leq 2M$. This establishes the assertion (i) in the Theorem.

To demonstrate assertion (ii) we form the following argument.

$\sum_{i\in H} |\inf\{p^* \cdot (x - r^i) | x \succ_i x^{\circ i}\}|$

$= \sum_{i\in H^+} |\inf\{p^* \cdot y | y \in \Gamma^i\}| + \sum_{i\in H^-} |\inf\{p^* \cdot y | y \in \Gamma^i\}|$

$\leq [-\sum_{i\in H^+} \inf\{p^* \cdot y | y \in \Gamma^i \cup \{0\}\} + \sum_{i\in H^+} p^* \cdot (x^{\circ i} - r^i)]$ $- \sum_{i\in H^-} \inf\{p^* \cdot y | y \in \Gamma^i \cup \{0\}\}$ (Using the inequality (**), the term in square brackets is larger than the first term of the previous expression and — taking account of signs — the last term exceeds the last term of the previous expression).

$= -\sum_{i\in H^+} \inf\{p^* \cdot y | y \in \Gamma^i \cup \{0\}\} - \sum_{i\in H^-} \inf\{p^* \cdot y | y \in \Gamma^i \cup \{0\}\} + \sum_{i\in H^+} p^* \cdot (x^{\circ i} - r^i)$ (Then using the identity at the start of this step, and the expression (*))

$= -\inf\{p^* \cdot y | y \in \Omega\} + \sum_{i\in H^+} p^* \cdot (x^{\circ i} - r^i)$

$\leq M + M = 2M$.

Thus $\sum_{i\in H} |\inf\{p^* \cdot (x - r^i) | x \succ_i x^{\circ i}\}| \leq 2M$

QED