

Economics 200B Prof. R. Starr UCSD Winter 2009

Lecture Notes for January 22, 2009

A market economyFirms, profits, and household income $H, F, \alpha^{ij} \in \mathbf{R}_+, \sum_{i \in H} \alpha^{ij} = 1,$

$$r \equiv \sum_{i \in H} r^i.$$

Theorem 6.1 Assume P.II, P.III, and P.VI. $\tilde{\pi}^j(p)$ is a well-defined continuous function of p for all $p \in \mathbf{R}_+^N, p \neq 0$. $\tilde{\pi}^j(p)$ is homogeneous of degree 1.

$$\tilde{M}^i(p) = p \cdot r^i + \sum_{j \in F} \alpha^{ij} \tilde{\pi}^j(p).$$

$$P = \left\{ p \mid p \in \mathbf{R}^N, p_k \geq 0, k = 1, \dots, N, \sum_{k=1}^N p_k = 1 \right\}.$$

Excess demand and Walras' Law

Definition The excess demand function at prices $p \in P$ is

$$\tilde{Z}(p) = \tilde{D}(p) - \tilde{S}(p) - r = \sum_{i \in H} \tilde{D}^i(p) - \sum_{j \in F} \tilde{S}^j(p) - \sum_{i \in H} r^i.$$

Lemma 6.1 Assume C.I–C.V, C.VI(SC), C.VII, P.II, P.III, P.V, and P.VI. The range of $\tilde{Z}(p)$ is bounded. $\tilde{Z}(p)$ is continuous and well defined for all $p \in P$.

Proof Apply Theorems 4.1, 5.2, and 6.1. The finite sum of bounded sets is bounded. The finite sum of continuous functions is continuous. QED

Theorem 6.2 (Weak Walras' Law) Assume C.I–C.V, C.VI(SC), C.VII, P.II, P.III, P.V, and P.VI. For all $p \in P$, $p \cdot \tilde{Z}(p) \leq 0$. For p such that $p \cdot \tilde{Z}(p) < 0$, there is $k = 1, 2, \dots, N$ so that $\tilde{Z}_k(p) > 0$.

Proof of Theorem 6.2 $p \cdot \tilde{D}^i(p) \leq \tilde{M}^i(p) = p \cdot r^i + \sum_{j \in F} \alpha^{ij} \tilde{\pi}^j(p)$. $\sum_{i \in H} \alpha^{ij} = 1$ for each $j \in F$.

$$p \cdot \tilde{Z}(p) = p \cdot \left[\sum_{i \in H} \tilde{D}^i(p) - \sum_{j \in F} \tilde{S}^j(p) - \sum_{i \in H} r^i \right]$$

$$\begin{aligned}
 &= p \cdot \sum_{i \in H} \tilde{D}^i(p) - p \cdot \sum_{j \in F} \tilde{S}^j(p) - p \cdot \sum_{i \in H} r^i \\
 &= \sum_{i \in H} p \cdot \tilde{D}^i(p) - \sum_{j \in F} p \cdot \tilde{S}^j(p) - \sum_{i \in H} p \cdot r^i \\
 &= \sum_{i \in H} p \cdot \tilde{D}^i(p) - \sum_{j \in F} \tilde{\pi}^j(p) - \sum_{i \in H} p \cdot r^i \\
 &= \sum_{i \in H} p \cdot \tilde{D}^i(p) - \sum_{j \in F} \left[\sum_{i \in H} \alpha^{ij} \tilde{\pi}^j(p) \right] - \sum_{i \in H} p \cdot r^i \\
 &= \sum_{i \in H} p \cdot \tilde{D}^i(p) - \sum_{i \in H} \left[\sum_{j \in F} \alpha^{ij} \tilde{\pi}^j(p) \right] - \sum_{i \in H} p \cdot r^i \\
 &\quad \text{Note the change in the order of summation} \\
 &= \sum_{i \in H} p \cdot \tilde{D}^i(p) - \sum_{i \in H} \left\{ \left[\sum_{j \in F} \alpha^{ij} \tilde{\pi}^j(p) \right] + p \cdot r^i \right\} \\
 &= \sum_{i \in H} p \cdot \tilde{D}^i(p) - \sum_{i \in H} \tilde{M}^i(p) \\
 &= \sum_{i \in H} \left[p \cdot \tilde{D}^i(p) - \tilde{M}^i(p) \right] \leq 0.
 \end{aligned}$$

since $p \cdot \tilde{D}^i(p) \leq \tilde{M}^i(p)$ This proves the weak inequality as required.

We now must demonstrate the positivity of some coordinate of $\tilde{Z}(p)$ when the strict inequality holds. Let $p \cdot \tilde{Z}(p) < 0$. Then $p \cdot \sum_{i \in H} \tilde{D}^i(p) < p \cdot r + p \cdot \sum_{j \in F} \tilde{S}^j(p) = \sum_{i \in H} \tilde{M}^i(p)$, so for some $i' \in H$, $p \cdot \tilde{D}^{i'}(p) < \tilde{M}^{i'}(p)$. Now we apply Lemma 5.3. We must have $|\tilde{D}^{i'}(p)| = c$. Recall that c is chosen so that $|x| < c$ (a strict inequality) for all attainable x . But then $\tilde{D}^{i'}(p)$ is not attainable. For no $y \in \mathcal{Y}$ do we have $\tilde{D}^{i'}(p) \leq y + r$. But for all $i \in H$, $\tilde{D}^i(p) \in \mathbf{R}_+^N$. So $\sum_{i \in H} \tilde{D}^i(p) \geq \tilde{D}^{i'}(p)$. Therefore, $\tilde{Z}_k(p) > 0$, for some $k = 1, 2, \dots, N$. QED

General equilibrium of the market economy with an excess demand function
Existence of equilibrium

$$P = \left\{ p \mid p \in \mathbf{R}^N, p_k \geq 0, k = 1, \dots, N, \sum_{k=1}^N p_k = 1 \right\}.$$

$$\tilde{Z}(p) = \sum_{i \in H} \tilde{D}^i(\cdot) - \sum_{j \in F} \tilde{S}^j(\cdot) - r.$$

Definition $p^\circ \in P$ is said to be an equilibrium price vector if $\tilde{Z}(p^\circ) \leq 0$ (the inequality holds coordinatewise) with $p_k^\circ = 0$ for k such that $\tilde{Z}_k(p^\circ) < 0$.

Weak Walras' Law (Theorem 6.2): For all $p \in P$, $p \cdot \tilde{Z}(p) \leq 0$. For p such that $p \cdot \tilde{Z}(p) < 0$, there is $k = 1, 2, \dots, N$ so that $\tilde{Z}_k(p) > 0$, under assumptions C.I–C.V, C.VI(SC), P.II, P.III, P.V, and P.VI.

Continuity: $\tilde{Z}(p)$ is a continuous function, assuming P.II, P.III, P.V, P.VI, C.I–C.V, C.VI(SC) and C.VII (Theorems 4.1, 5.2, and 6.1).

Theorem 2.10 Brouwer Fixed-Point Theorem: Let S be an N -simplex and let $f : S \rightarrow S$, where f is continuous. Then there is $x^* \in S$ so that $f(x^*) = x^*$.

Theorem 7.1 Assume P.II, P.III, P.V, P.VI, C.I–C.V, C.VI (SC), and C.VII. There is $p^* \in P$ so that p^* is an equilibrium.

Proof $T : P \rightarrow P$

For $k = 1, 2, 3, \dots, N$:

$$T_k(p) \equiv \frac{p_k + \max[0, \tilde{Z}_k(p)]}{1 + \sum_{n=1}^N \max[0, \tilde{Z}_n(p)]} = \frac{p_k + \max[0, \tilde{Z}_k(p)]}{\sum_{n=1}^N \{p_n + \max[0, \tilde{Z}_n(p)]\}}$$

By Lemma 6.1, $\tilde{Z}(p)$ is a continuous function. Then $T(p)$ is a continuous function from the simplex into itself. By the Brouwer Fixed-Point Theorem there is $p^* \in P$ so that $T(p^*) = p^*$. But then for all $k = 1, \dots, N$,

$$T_k(p^*) = p_k^* = \frac{p_k^* + \max[0, \tilde{Z}_k(p^*)]}{1 + \sum_{n=1}^N \max[0, \tilde{Z}_n(p^*)]}$$

To avoid repeated tedious notation, let

$$0 < \alpha = \frac{1}{1 + \sum_{n=1}^N \max[0, \tilde{Z}_n(p^*)]} \leq 1.$$

We'll demonstrate that $\tilde{Z}_n(p^*) \leq 0$ all n .¹

¹ In the case $\alpha = 1$, trivially $\tilde{Z}_n(p^*) \leq 0$ all n , and we have only to show that $p_k^* = 0$ when $\tilde{Z}_k(p^*) < 0$.

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We have

$$T_k(p^*) = p_k^* = \alpha(p_k^* + \max[0, \tilde{Z}_k(p^*)])$$

$$p_k^* = \alpha p_k^* + \alpha \max[0, \tilde{Z}_k(p^*)]$$

or

$$(1 - \alpha)p_k^* = \alpha \max[0, \tilde{Z}_k(p^*)].$$

Multiplying through by $\tilde{Z}_k(p^*)$, we get

$$(1 - \alpha)p_k^* \tilde{Z}_k(p^*) = \alpha(\max[0, \tilde{Z}_k(p^*)]) \tilde{Z}_k(p^*) \quad (*).$$

We can restate the Weak Walras' Law as

$$0 \geq p^* \cdot \tilde{Z}(p^*) = \sum_{k=1}^N p_k^* \tilde{Z}_k(p^*).$$

Multiplying through by $(1 - \alpha)$, and substituting per (*) we get

$$0 \geq (1 - \alpha)p^* \cdot \tilde{Z}(p^*) = \sum_{k=1}^N (1 - \alpha)p_k^* \tilde{Z}_k(p^*) = \alpha \sum_{k=1}^N (\max[0, \tilde{Z}_k(p^*)]) \tilde{Z}_k(p^*).$$

Then the sum on the right-hand side is ≤ 0 , but it would be strictly positive if there were any k so that $\tilde{Z}_k(p^*) > 0$, generating a squared term in $\tilde{Z}_k(p^*)$.

But this means that $\tilde{Z}_k(p^*) \leq 0$, for all k . Then there is no k so that $\tilde{Z}_k(p^*) > 0$. From the Weak Walras' Law it follows that we cannot have $p^* \cdot \tilde{Z}(p^*) < 0$, so it follows that $p^* \cdot \tilde{Z}(p^*) = 0$. Hence for k so that $\tilde{Z}_k(p^*) < 0$, we have $p_k^* = 0$. This completes the proof. QED

Lemma 7.1 Assume P.II, P.III, P.V, P.VI, C.I–C.V, C.VI(SC), and C.VII. Let p^* be an equilibrium. Then for all $i \in H$, $|\tilde{D}^i(p^*)| < c$, where c is the bound on the Euclidean length of demand, $\tilde{D}^i(p^*)$. Further, in equilibrium, Walras' Law holds as an equality: $p^* \cdot \tilde{Z}(p^*) = 0$.

Proof Since $\tilde{Z}(p^*) \leq 0$ (coordinatewise), we know that

$$\sum_{i \in H} \tilde{D}^i(p^*) \leq \sum_{j \in F} \tilde{S}^j(p^*) + \sum_{i \in H} r^i,$$

where the inequality holds coordinatewise. However, that implies that the aggregate consumption $\sum_{i \in H} \tilde{D}^i(p^*)$ is attainable, so for each household i , $|\tilde{D}^i(p^*)| < c$, where c is the bound on demand, $\tilde{D}^i(\cdot)$.

We have for all p , $p \cdot \tilde{Z}(p) \leq 0$. In equilibrium, at p^* , we have $\tilde{Z}(p^*) \leq 0$ (coordinatewise) with $p_k^* = 0$ for k so that $\tilde{Z}_k(p^*) < 0$. Therefore $p^* \cdot \tilde{Z}(p^*) = 0$. QED