Economics 200B Prof. R. Starr UCSD Winter 2009

Lecture Notes for January 20, 2009

Households

5.1 The structure of household consumption sets and preferences

Households are elements of the finite set H numbered $1, 2, \ldots, \#H$. A household $i \in H$ will be characterized by its possible consumption set $X^i \subseteq \mathbf{R}^N_+$, its preferences $\succeq_i$, and its endowment $r^i \in \mathbf{R}^N_+$.

5.1.1 Consumption sets

(C.I) X^i is closed and nonempty.
(C.II) $X^i \subseteq \mathbf{R}^N_+$. X^i is unbounded above, that is, for any $x \in X^i$ there is $y \in X^i$ so that $y > x$, that is, for $n = 1, 2, ..., N, y_n \geq x_n$ and $y \neq x$.
(C.III) X^i is convex.

It is usually simplest to take X^i to be the nonnegative orthant (quadrant) of $\mathbf{R}^N$, denoted $\mathbf{R}^N_+$. We will take the possible aggregate (for the economy's household sector) consumption set to be $X = \sum_{i \in H} X^i$.

5.1.2 Preferences

Each household $i \in H$ has a preference quasi-ordering on X^i, denoted $\succeq_i$. For typical $x, y \in X^i$, "$x \succeq_i y$" is read "x is preferred or indifferent to y (according to i)." We introduce the following terminology:

If $x \succeq_i y$ and $y \succeq_i x$ then $x \sim_i y$ ("x is indifferent to y"),
If $x \succeq_i y$ but not $y \succeq_i x$ then $x \succ_i y$ ("x is strictly preferred to y").

We will assume $\succeq_i$ to be complete on X^i, that is, any two elements of X^i are comparable under $\succeq_i$. For all $x, y \in X^i$, $x \succeq_i y$, or $y \succeq_i x$ (or both). Since we take $\succeq_i$ to be a quasi-ordering, $\succeq_i$ is assumed to be transitive and reflexive.

utility function $u^i(x)$ so that $x \succeq_i y$ if and only if $u^i(x) \geq u^i(y)$.
Just read $u^i(x) \geq u^i(y)$ wherever you see $x \succeq_i y$.

5.1.3 Non-Satiation

(C.IV) (Non-Satiation) Let $x \in X^i$. Then there is $y \in X^i$ so that $y \succ_i x$.

Occasional stronger alternative

(C.IV**) (Weak Desirability) X^i contains a translation of R_+^N. $x, y \in X^i, x >> y$(i.e. $x_n > y_n,$ for all n) implies $x \succ_i y$.

5.1.4 Continuity

(C.V) (Continuity) For every $x^\circ \in X^i$, the sets $A^i(x^\circ) = \{x \mid x \in X^i, x \succeq_i x^\circ\}$ and $G^i(x^\circ) = \{x \mid x \in X^i, x^\circ \succeq_i x\}$ are closed.

The structure of the upper and lower contour sets of $\succeq_i$ assumed in C.V is precisely the behavior we'd expect if $\succeq_i$ were defined by a continuous utility function. This follows since the inverse image of a closed set under a continuous mapping is closed (Theorem 2.6).

Example 5.1 (Lexicographic preferences) The lexicographic (dictionary-like) ordering on $\mathbf{R}^N$ (let's denote it $\succeq_L$) is described in the following way. Let $x = (x_1, x_2, \ldots, x_N)$ and $y = (y_1, y_2, \ldots, y_N)$.

$x \succ_L y$if $x_1 > y_1$, or
if $x_1 = y_1$ and $x_2 > y_2$, or
if $x_1 = y_1$, $x_2 = y_2$, and $x_3 > y_3$, and so forth ....
$x \sim_L y$if $x = y$.

$\succeq_L$ fulfills non-satiation, trivially fulfills strict convexity , but does not fulfill continuity (C.V).

5.1.5 Attainable Consumption

Definition x is an ***attainable*** consumption if $y + r \geq x \geq 0$, where $y \in \mathcal{Y}$ and $r \in \mathbf{R}_+^N$ is the economy's initial resource endowment, so that y is an attainable production plan.

Note that the set of attainable consumptions is bounded under P.VI.

5.1.6 Convexity of preferences

(C.VI)(WC) (Weak Convexity of Preferences) $x \succeq_i y$ implies $((1-\alpha)x+\alpha y) \succeq_i y$, for $0 \leq \alpha \leq 1$.

(C.VI)(SSC) (Semi-strict convexity of Preferences) $x \succ_i y$ implies $((1-\alpha)x + \alpha y) \succ_i y$, for $0 \leq \alpha < 1$.

(C.VI)(SC) (Strict Convexity of Preferences): Let $x \succeq_i y$, (note that this includes $x \sim_i y$), $x \neq y$, and let $0 < \alpha < 1$. Then

$$\alpha x + (1-\alpha)y \succ_i y.$$

Equivalently, if preferences are characterized by a utility function $u^i(\bullet)$, then we can state C.VI(SC) as

$$u^i(x) \geq u^i(y), x \neq y, \text{ implies } u^i[\alpha x + (1-\alpha)y] > u^i(y).$$

An immediate consequence of C.V and C.VI(WC) is that $A^i(x^\circ)$ is convex for every $x^\circ \in X^i$.

5.2 Representation of $\succeq$: Existence of a continuous utility function

Definition Let u^i: $X^i \to \mathbf{R}$. $u^i(\bullet)$ is a utility function that ***represents*** the preference ordering $\succeq_i$ if for all $x, y \in X^i$, $u^i(x) \geq u^i(y)$ if and only if $x \succeq_i y$. This implies that $u^i(x) > u^i(y)$ if and only if $x \succ_i y$.

The function $u^i(\bullet)$, i's utility function, is merely a representation of i's preference ordering $\succeq_i$; $u^i(\bullet)$ contains no additional information. In particular, it does not represent strength or intensity of preference. Utility functions like $u^i(\cdot)$ that represent an ordering $\succeq_i$, without embodying additional information or assumptions, are called ordinal (i.e., representing an ordering). In this sense, any monotone (order-preserving) transformation of $u^i(\cdot)$, $v^i(\cdot)$, is equally appropriate as a representation of $\succeq_i$.

5.2.1 Construction of a continuous utility function

5.2.1.1 Weak Monotonicity

Example 5.2 Assume C.I - C.III, C.IV**, C.V. Let X^i include a translation of R^N_+ (this includes the special case where $X^i = \mathrm{R}^N_+$. (Weak Monotonicity) Let $x, y \in X^i$ and $x >> y (x_n > y_n, n = 1, 2, ..., N)$. Then $x \succ_i y$. Weak monotonicity is a strong form of nonsatiation, C.IV. Then there is a utility function $u^i(\cdot)$ continuous throughout X^i.

In this case it is easy to construct a continuous utility function representing $\succeq_i$. Just draw the 45° ray from the origin in R^N_+ and let the utility value of each point be the length of the ray where the indifference curve through the point intersects the ray.

5.2.1.2 A bounded domain

Example 5.3 Assume C.I - C.V, C.VI (SSC). Let $\mathcal{S}$ be a compact convex subset of R^N, so that $X^i \cap \mathcal{S} \neq \emptyset$. Then there is $u^i : [X^i \cap \mathcal{S}] \to \mathrm{R}$ so that

$u^i(\cdot)$ represents $\succeq_i$ on $X^i \cap \mathcal{S}$ and u^i is continuous on $X^i \cap \mathcal{S}$.

Proof The approach to demonstrating this example is similar to the demonstration of the previous example.

We seek two points: w°, the least desirable point in $X^i \cap \mathcal{S}$, and z°, the most desirable point.

Let $w^\circ \in \bigcap_{w \in X^i \cap \mathcal{S}}[G^i(w) \cap \mathcal{S}]$. By the nested intervals property w° exists. w° is the least desirable point in $X^i \cap \mathcal{S}$.

We'll find z° in the same way.

Let $z^\circ \in \bigcap_{z \in X^i \cap \mathcal{S}}[A^i(z) \cap \mathcal{S}]$. By the nested intervals property z° exists. z° is the most desirable point in $X^i \cap \mathcal{S}$.

If $w^\circ = z^\circ$ the example is trivially satisfied. Suppose w° and z° are distinct.

Define the chord between w° and z° as

$\mathcal{L} \equiv \{x \in X^i \cap \mathcal{S} | x = (1-\alpha)w^\circ + \alpha z^\circ, 0 \leq \alpha \leq 1\}$.

Note that $\mathcal{L} \in X^i \cap \mathcal{S}$ by convexity of X^i and $\mathcal{S}$. We now define $u^i(x)$ in the following way. Let $x \in X^i \cap \mathcal{S}$. Recall the upper and lower contour preference sets $A^i(x)$ and $G^i(x)$. Let y^* be the point at the intersection,

$$\{y^*\} = \mathcal{L} \cap A^i(x) \cap G^i(x).$$

We know this intersection is nonempty since X^i is closed and connected (by C.III, convexity) and because $A^i(x)$ and $G^i(x)$ are closed and nonempty and their union equals X^i (by completeness of $\succeq_i$). That is, by connectedness of X^i, $A^i(x)$ and $G^i(x)$ cannot be disjoint (even along $\mathcal{L}$). By semi-strict convexity of $\succeq_i$, C.VI(SSC), there is only a single point in this intersection. For each $x \in X^i$, let

$$u^i(x) \equiv |y^* - w^\circ|,$$

where y^* is defined as above. That is, we define $u^i(x)$ to be the Euclidean length of the ray along $\mathcal{L}$ from w° to a point in $\mathcal{L}$ indifferent to x. Then $u^i(x)$ is well defined. QED

5.3 Choice and boundedness of budget sets, $\tilde{\boldsymbol{B}}$ $(\boldsymbol{p})$

$\tilde{B}^i(p)$, Recall that x is an ***attainable*** consumption if $y + r \geq x \geq 0$, where $y \in \mathcal{Y}$ and $r \in \mathbf{R}_+^N$ is the economy's initial resource endowment, so that y is an attainable production plan. The set of attainable consumptions is bounded under P.VI.

Choose $c \in \mathbf{R}_+$ so that $|x| < c$ (a strict inequality) for all attainable consumptions x. Choose c sufficiently large that $X^i \cap \{x \mid x \in \mathbf{R}^N, c > |x|\} \neq \phi$.

We assign to household i, a budget at prices p of $\tilde{M}^i(p)$.

Let

$$\tilde{B}^i(p) = \{x \mid x \in \mathbf{R}^N, p \cdot x \leq \tilde{M}^i(p)\} \cap \{x||x| \leq c\}.$$

$$\begin{aligned}\tilde{D}^i(p) &\equiv \{x \mid x \in \tilde{B}^i(p) \cap X^i, x \succeq_i y \text{ for all } y \in \tilde{B}^i(p) \cap X^i\} \\ &\equiv \{x \mid x \in \tilde{B}^i(p) \cap X^i, x \text{ maximizes } u^i(y) \text{ for all } y \in \tilde{B}^i(p) \cap X^i\}.\end{aligned}$$

$$\tilde{D}(p) = \sum_{i \in H} \tilde{D}^i(p).$$

Lemma 5.1 $\tilde{B}^i(p)$ is a closed set.

Lemma 5.2 Let $\tilde{M}^i(p)$ be homogeneous of degree 1. Let $\tilde{B}^i(p)$ and $\tilde{D}^i(p)$ $\neq \emptyset$. Then $\tilde{B}^i(p)$ and $\tilde{D}^i(p)$ are homogeneous of degree 0.

$$P \equiv \left\{p \mid p \in \mathbf{R}^N, p_n \geq 0, n = 1, 2, 3, \ldots, N, \sum_{n=1}^{N} p_n = 1\right\}.$$

5.3.1 Adequacy of income

(C.VII) For all $i \in H$, $\tilde{M}^i(p) > \inf_{x \in X^i \cap \{x \in R^N, |x| \leq c\}} p \cdot x$ for all $p \in P$.

Example 5.4 (The Arrow Corner)

$$\begin{aligned}X^i &= \mathbf{R}^2_+, \\ r^i &= (1, 0), \\ \tilde{M}^i(p) &= p \cdot r^i.\end{aligned}$$

Let $p^\circ = (0, 1)$. Then

$$\tilde{B}^i(p^\circ) \cap X^i = \{(x, y) \mid c \geq x \geq 0, y = 0\},$$

the truncated nonnegative x axis. Consider the sequence $p^\nu = (1/\nu, 1 - 1/\nu)$. $p^\nu \to p^\circ$. We have

$$\tilde{B}^i(p^\nu) \cap X^i = \left\{(x, y) \mid p^\nu \cdot (x, y) \leq \frac{1}{\nu}, (x, y) \geq 0, c \geq |(x, y)| \geq 0\right\},$$

$(c,0)\in\tilde{B}^i(p^\circ)$, but there is no sequence $(x^\nu, y^\nu)\in\tilde{B}^i(p^\nu)$ so that $(x^\nu, y^\nu) \to (c,0)$. On the contrary, for any sequence $(x^\nu, y^\nu) \in \tilde{B}^i(p^\nu)$ so that $(x^\nu, y^\nu) = \tilde{D}^i(p^\nu)$, (x^ν, y^ν) will converge to some $(x^*, 0)$, where $0 \le x^* \le 1$. For suitably chosen $\succeq_i$, we may have $(c,0) = \tilde{D}^i(p^\circ)$. Hence $\tilde{D}^i(p)$ need not be continuous at p°. This completes the example.

5.4 Demand behavior under strict convexity

Theorem 5.2 Assume C.I–C.V, C.VI(SC), and C.VII. Let $\tilde{M}^i(p)$ be a continuous function for all $p \in P$. Then $\tilde{D}^i(p)$ is a well-defined, point-valued, continuous function for all $p \in P$.

Proof $\tilde{B}^i(p)\cap X^i$ is the intersection of the closed set $\{x \mid p\cdot x \le \tilde{M}^i(p)\}$ with the compact set $\{x \mid |x| \le c\}$ and the closed set X^i. Hence it is compact. It is nonempty by C.VII. Because $\tilde{D}^i(p)$ is characterized by the maximization of a continuous function, $u^i(\cdot)$, on this compact nonempty set, there is a well-defined maximum value, $u^* = u^i(x^*)$, where x^* is the utility-optimizing value of x in $\tilde{B}^i(p) \cap X^i$. We must show that x^* is unique for each $p \in P$ and that x^* is a continuous function of p.

We will now demonstrate that uniqueness follows from strict convexity of preferences (C.VI(SC)). Suppose there is $x' \in \tilde{B}^i(p) \cap X^i, x' \ne x^*, x' \sim_i x^*$. We must show that this leads to a contradiction. But now consider a convex combination of x' and x^*. Choose $0 < \alpha < 1$. The point $\alpha x' + (1-\alpha)x^* \in \tilde{B}^i(p) \cap X^i$ by convexity of X^i and $\tilde{B}^i(p)$. But C.VI(SC), strict convexity of preferences, implies that $[\alpha x' + (1-\alpha)x^*] \succ_i x' \sim_i x^*$. This is a contradiction, since x^* and x' are elements of $\tilde{D}^i(p)$. Hence x^* is the unique element of $\tilde{D}^i(p)$. We can now, without loss of generality, refer to $\tilde{D}^i(p)$ as a (point-valued) function.

To demonstrate continuity, let $p^\nu \in P$, $\nu = 1, 2, 3, \ldots, p^\nu \to p^\circ$. We must show that $\tilde{D}^i(p^\nu) \to \tilde{D}^i(p^\circ)$. $\tilde{D}^i(p^\nu)$ is a sequence in a compact set. Without loss of generality take a convergent subsequence, $\tilde{D}^i(p^\nu) \to x^\circ$. We must show that $x^\circ = \tilde{D}^i(p^\circ)$. We will use a proof by contradiction.

Define

$$\hat{x} = \underset{x\in X^i\cap\{y|y\in\mathbf{R}^N, c\ge|y|\}}{\arg\min} p^\circ \cdot x.$$

The expression "$\hat{x} = \arg\min_{x\in X^i\cap\{y|y\in\mathbf{R}^N, c\ge|y|\}} p^0 \cdot x$" defines $\hat{x}$ as the minimizer of $p^\circ \cdot x$ in the domain $X^i \cap \{y \mid y \in \mathbf{R}^N, c \ge |y|\}$. $\hat{x}$ is well defined (though it may not be unique) since it represents a minimum of a continuous function taken over a compact domain.

Now consider two cases. In each case we will construct a sequence w^ν in $X^i \cap \{y \mid y \in \mathbf{R}^N, c \geq |y|\}$.

Case 1: If $p^\circ \cdot \tilde{D}^i(p^\circ) < \tilde{M}^i(p^\circ)$ for ν large $p^\nu \cdot \tilde{D}^i(p^\circ) < \tilde{M}^i(p^\nu)$. Then let $w^\nu = D^i(p^\circ)$.

Case 2: If $p^\circ \cdot \tilde{D}^i(p^\circ) = \tilde{M}^i(p^\circ)$ then by (C.VII) $p^\circ \cdot \tilde{D}^i(p^\circ) > p^\circ \cdot \hat{x}$.

Let

$$\alpha^\nu = \min\left[1, \frac{\tilde{M}^i(p^\nu) - p^\nu \cdot \hat{x}}{p^\nu \cdot (\tilde{D}^i(p^\circ) - \hat{x})}\right].$$

For ν large, the denominator is positive, α^ν is well defined (this is where C.VII enters the proof), and $0 \leq \alpha^\nu \leq 1$. Let $w^\nu = (1-\alpha^\nu)\hat{x} + \alpha^\nu \tilde{D}^i(p^\circ)$. Note that $\tilde{M}^i(p)$ is continuous in p. The fraction in the definition of α^ν is the proportion of the move from $\hat{x}$ to $\tilde{D}^i(p^\circ)$ that the household can afford at prices p^ν. As ν becomes large, the proportion approaches or exceeds unity.

Then in both Case 1 and Case 2, $w^\nu \to \tilde{D}^i(p^\circ)$ and $w^\nu \in \tilde{B}^i(p^\nu) \cap X^i$. Suppose, contrary to the theorem, $x^\circ \neq \tilde{D}^i(p^\circ)$. Then $u^i(x^\circ) < u^i(\tilde{D}^i(p^\circ))$. But u^i is continuous, so $u^i(\tilde{D}^i(p^\nu) \to u^i(x^\circ)$ and $u^i(w^\nu) \to u^i(\tilde{D}^i(p^\circ))$. Thus, for ν large, $u^i(w^\nu) > u^i(\tilde{D}^i(p^\nu))$. But this is a contradiction, since $\tilde{D}^i(p^\nu)$ maximizes $u^i(\cdot)$ in $\tilde{B}^i(p^\nu) \cap X^i$. The contradiction proves the result. This completes the demonstration of continuity. QED

Lemma 5.3 Assume C.I–C.V, C.VI(SSC), and C.VII. Then $p \cdot \tilde{D}^i(p) \leq \tilde{M}^i(p)$. Further, if $p \cdot \tilde{D}^i(p) < \tilde{M}^i(p)$ then $|\tilde{D}^i(p)| = c$.

Proof $\tilde{D}^i(p) \in \tilde{B}^i(p)$ by definition. However, that ensures $p \cdot \tilde{D}^i(p) \leq \tilde{M}^i(p)$ and hence the weak inequality surely holds. Suppose, however, $p \cdot \tilde{D}^i(p) < \tilde{M}^i(p)$ and $|\tilde{D}^i(p)| < c$. We wish to show that this leads to a contradiction. Recall C.IV (Non-Satiation) and C.VI(SSC) (Semi-Strict Convexity). By C.IV there is $w^* \in X^i$ so that $w^* \succ_i \tilde{D}^i(p)$. Clearly, $w^* \notin \tilde{B}^i(p)$ so one (or both) of two conditions holds: (a) $p \cdot w^* > \tilde{M}^i(p)$, (b) $|w^*| > c$.

Set $w' = \alpha w^* + (1-\alpha)\tilde{D}^i(p)$. There is an $\alpha(1 > \alpha > 0)$ sufficiently small so that $p \cdot w' \leq \tilde{M}^i(p)$ and $|w'| \leq c$. Thus $w' \in \tilde{B}^i(p)$. Now $w' \succ_i \tilde{D}^i(p)$ by C.VI(SSC), which is a contradiction since $\tilde{D}^i(p)$ is supposed to be the preference optimizer in $\tilde{B}^i(p)$. The contradiction shows that we cannot have both $p \cdot \tilde{D}^i(p) < \tilde{M}^i(p)$ and $|\tilde{D}^i(p)| < c$. Hence, if the first inequality holds, we must have $|\tilde{D}^i(p)| = c$. QED