Lecture Notes for February 14, 2008

Happy Valentine's Day!!

Recall the Separating Hyperplane Theorem: Let $A, B \subset \mathbf{R}^N$; let A and B be nonempty, convex, and disjoint, that is $A \cap B = \phi$. Then there is $p \in \mathbf{R}^N, p \neq 0$, so that $p \cdot x \geq p \cdot y$, for all $x \in A, y \in B$.

1 The Shapley-Folkman Theorem

Shapley-Folkman Theorem tells us that the sum of a large number of nonconvex sets — though still nonconvex — is approximately convex. The nonconvexities do not compound each other indefinitely.

1.1 Nonconvex sets and their convex hulls

The *convex hull* of a set S will be the smallest convex set containing S. The convex hull of S will be denoted *con(S)*. We can define *con(S)* , for $S \subset R^N$ as follows

$$\mathit{con(S)} \equiv \{x \mid x = \sum_{i=0}^{N} \alpha^i x^i, \text{where } x^i \in S, \alpha^i \geq 0 \text{ all i, and } \sum_{i=0}^{N} \alpha^i = 1\}$$

or equivalently as

$$\mathit{con(S)} \equiv \bigcap_{S \subset\ T \subset R^N; T\ \mathit{convex}} T.$$

1.2 The Shapley-Folkman Lemma

Lemma (Shapley-Folkman): Let S^1, S^2, S^3, $\ldots$,S^m, be nonempty compact subsets of R^N. Let $x \in con(S^1 + S^2 + S^3 + \ldots + S^m)$. Then for each i=1,2,...,m, there is $y^i \in con(S^i)$ so that $\sum_{i=1}^m y^i = x$ and with at most N exceptions, $y^i \in S^i$. Equivalently: Let F be a finite family of nonempty compact sets in R^N and let $y \in con(\sum_{S \in F} S)$. Then there is a partition of F into two disjoint subfamilies F' and F'' with the number of elements in $F' \leq N$ so that $y \in \sum_{S \in F'} con(S) + \sum_{S \in F''} S$.

Simple example: Let $S^i = \{(0,0), (0,1), (1,0), (1,1)\}$ for i = 1, 2, ..., 10. $con(S^1 + S^2 + S^3 + \ldots + S^{10}) = \{x | x \in R^2,\ 0 \leq x_1, x_2 \leq 10\}$. Choose a typical point in $con(S^1 + S^2 + S^3 + \ldots + S^{10})$, say $x = (5.5, 5.7)$. The lemma says that x can be represented as a sum of points in the convex hulls of the original sets, $con(S^1)$, $con(S^2)$, $\ldots$, $con(S^{10})$.

More important, the theorem says that x can be represented in this way as as sum of points most (all but two in R^2) coming from the original sets S^1, S^2, S^3, $\ldots S^{10}$, not from points of their convex hulls that were not part of the original sets S^i. For example, Let $x^1 = (0.5, 0) \in con(S^1)$, $x^2 = (0, 0.7) \in con(S^2)$, $x^3 = (1, 1) \in S^3$, $x^4 = (1, 1) \in S^4$, $x^5 = (1, 1) \in S^5$, $x^6 = (1, 1) \in S^6$, $x^7 = (1, 1) \in S^7$, $x^8 = (0, 0) \in S^8$, $x^9 = (0, 0) \in S^9$, $x^{10} = (0, 0) \in S^{10}$. Then $x = \sum_{i=1}^{10} x^i$, all $x^i \in con(S^i)$ and with only two exceptions $x^i \in S^i$. This is just what the Shapley-Folkman Lemma asserts.

1.3 Measuring Non-Convexity, The Shapley-Folkman Theorem

Definition: The radius of a compact set S is defined as
$rad(S) \equiv \inf_{x \in R^N} \sup_{y \in S} |x - y|$.

That is, the $rad(S)$ is the radius of the smallest closed ball centered in $con(S)$ containing S. This leads to

Theorem (Shapley - Folkman): Let F be a finite family of compact subsets $S \subset R^N$ and $L > 0$ so that $rad(S) \leq L$ for all $S \in F$. Then for any $x \in con(\sum_{S \in F} S)$ there is $y \in \sum_{S \in F} S$ so that $|x - y| \leq L\sqrt{N}$.

1.4 Corollary: A tighter bound

Definition: We define the inner radius of $S \subset R^N$ as
$r(S) \equiv \sup_{x \in con(S)} \inf_{T \subset S; x \in con(T)} rad(T)$

Corollary to the Shapley-Folkman Theorem: Let F be a finite family of compact subsets $S \subset R^N$ and $L > 0$ so that $r(S) \leq L$ for all $S \in F$. Then for any $x \in con(\sum_{S \in F} S)$ there is $y \in \sum_{S \in F} S$ so that $|x - y| \leq L\sqrt{N}$.

2 A Large Economy without Replication

Though the core convergence result for a replicated economy is very intuitive, it treats a specialized case — an economy becoming large through replication. It suggests a more general result: almost any large economy with a nonempty core should have its core close to competitive equilibrium. In fact, this result is true. We'll prove it using the Shapley-Folkman Lemma.

Recall the proof in the previous section. The core allocation is shown to be close to competitive equilibrium by showing that the set of preferred net trades is a convex set with the zero vector, 0, on the boundary, and running a supporting hyperplane through 0. Convexity is assured by filling in nonconvexities through replication. Then the normal to the supporting hyperplane, p, is the required competitive equilibrium price vector. The argument without replication follows the same logic, but it cannot fill in the the nonconvexities through replication. Rather, we use the Shapley-Folkman Lemma to show that the nonconvexities are of bounded size, small as a proportion of the number of households as that number becomes large.

Recall that the Shapley-Folkman Lemma says that the difference between a sum of sets and the convex hull of the sum is no larger than the N largest summands. In the present argument, we again form the set of preferred net trades and its convex hull. How far is the convex hull of the preferred net trade set from 0? No farther than the N largest summands. Then we can run a supporting hyperplane for this convex hull through a point offset from 0 by the N largest summands. How far is it from supporting the preferred net trade set? No farther than the N largest summands. Thus the normal to the supporting hyperplane supports the core allocation with a discrepancy fixed in size independent of the number of summands. As the economy becomes large, the discrepancy, per head of population, converges to 0.

We start by measuring the largest of the individual endowments. Define

$$M \equiv \max\{\sum_{h\in S} r_i^h | i = 1, ..., N, S \subseteq H, \#S = N\}$$

Trivially, M exists and is finite. We'll discuss M as though it is independent of the size of H. M is the largest amount of any of the N goods that any N-member subset of H can accumulate out of initial endowment. Then the N-dimensional vector $(M, M, ..., M)$ is an upper bound on the size of the sum of the endowments of any N-member coalition.

We will then prove the principal theorem of this section.

Theorem 14.3: Assume C.IV (weak monotonicity)for all h ∈ H, a pure exchange economy. Let $\{x^{oh}|h \in H\}$ be a core allocation for H. Then there is $p \in P$ so that

(i) $\sum_{h\in H} |p \cdot (x^{oh} - r^h)| \leq 2M$

(ii) $\sum_{h\in H} |\inf\{p \cdot (x - r^h)|x \succ_h x^{oh}\}| \leq 2M$

The theorem says that the core allocation $\{x^{oh}|h \in H\}$ is approximately a competitive equilibrium. Expression (i) expresses the approximation by saying that households approximately fulfill budget constraint at the core allocation. How close is the approximation? With prices on the unit simplex, the total by which households may under- and over-spend their budgets is 2M.

Expression (ii) says the core allocation nearly minimizes expenditure subject to utility constraint (equally satisfactory to $\{x^{oh}|h \in H\}$). How good is the approximation? Within 2M.

2M may be large. Nevertheless, we take M (and 2M) to be fixed by the character of the population. Then for a large economy, #H large, the ratio $2M/\#H$ is small. In the limit, as #H becomes arbitrarily large, $2M/\#H$ approaches zero. In a large economy, the typical discrepancy from competitive equilibrium becomes negligible.

Proof: Define Γ^i as in the proof of Theorem 14.2. $\Gamma^i = \{z \mid z \in \mathbf{R}^N, z+r^i \succ_i x^{oi}\}$. Define $\Omega \equiv \sum_{h\in H}\{\Gamma^h \cup \{0\}\}$.

The proof proceeds in several steps.

Step 1: Let R^N_{++} denote the strictly positive quadrant of R^N, that is, the interior of R^N_+. We claim $(-R^N_{++}) \cap \Omega = \emptyset$. The reason is straightforward. If there is a nonempty intersection we can form a blocking coalition and block the core allocation — but of course, the core is unblocked, so this leads to a contradiction.

Suppose contrary to the claim there is $z \in \Omega$ so that $z << 0$. Then there is $z^i \in \{\Gamma^i \cup \{0\}\}$ for each $i \in H$ so that $\sum_{i\in H} z^i << 0$. Take the subset $S \subset H$ of $i \in H$ corresponding to the nonzero elements z^i in this sum. Then for $i \in S$ there is $z^i \in \Gamma^i$ so that $\sum_{i\in S} z^i < 0$ (the inequality holds co-ordinatewise). But then S is a blocking coalition. That is for all $i \in S$, $z^i = x'^i - r^i$ so that $x'^i \succ_i x^{oi}$ and $\sum_{i\in S} x'^i \leq \sum_{i\in S} r^i$. This is a contradiction. Hence we have $(-R^N_{++}) \cap \Omega = \emptyset$ as claimed.

Step 2: Recall that the notation $con(A)$ denotes the convex hull of the set A. Define the set Z as the strictly negative quadrant of R^N translated to the southeast by M in each co-ordinate. That is, let

$Z \equiv \{z \in R^N | z_n < -M, \text{ for } n = 1, 2, ..., N\}$. In this step, we establish that $Z \cap con(\Omega) = \emptyset$.

Again, we use a proof by contradiction, establishing a blocking coalition in the event that the step were not fulfilled. Suppose contrary to the step, we have $Z \cap con(\Omega) \neq \emptyset$. Choose $z \in Z \cap con(\Omega)$. Then by the Shapley-Folkman Lemma we can represent z in the following way. There is a partition of H into disjoint subsets S and T with no more than N elements in T. There is a choice of $z^i \in con(\{\Gamma^i \cup \{0\}\})$ so that $z = \sum_{i\in S} z^i + \sum_{i\in T} z^i$, where for all $i \in S$, $z^i \in \{\Gamma^i \cup \{0\}\}$ and for all $i \in T, z^i \in [con(\{\Gamma^i \cup \{0\}\}) \backslash \{\Gamma^i \cup \{0\}\}]$. That is, a point in the convex hull of Ω is the sum of points of $con(\{\Gamma^i \cup \{0\}\})$ no more than N of which are from $[con(\{\Gamma^i \cup \{0\}\}) \backslash \{\Gamma^i \cup \{0\}\}]$. That is, most of the summands making up the convex hull of the sum will be from the original sets of the sum while a fixed finite number will be from the corresponding convex hulls. The original sum was nearly convex on its own.

Recall that for each i, $0 \in \{\Gamma^i \cup \{0\}\}$ and that $z << -(M, M, ..., M)$. Then the sum

$[\sum_{i\in S} z^i + \sum_{i\in T} 0] \in \Omega$. Note that each element of $con(\Gamma^i \cup \{0\}) \geq -r^i$ (the inequality applies co-ordinatewise). Then we have $[\sum_{i\in S} z^i + \sum_{i\in T} 0] = z - \sum_{i\in T} z^i \leq z + \sum_{i\in T} r^i << -(M, M, ..., M) + \sum_{i\in T} r^i \leq 0$. But then $(-R^N_{++}) \cap \Omega \neq \emptyset$ contradicting Step 1. The contradiction suffices to establish Step 2.

Step 3: By the Separating Hyperplane Theorem, there is $p^* \geq 0, p^* \neq 0$ and real k so that $p^* \cdot x \geq k \geq p^* \cdot y$ for all $x \in con(\Omega), y \in Z$. Then without loss of generality we take $p^* \in P$.

Step 4: $(x^{oh} - r^h) \in \bar{\Gamma}^h$ (the closure of Γ^h) so $p^* \cdot (x^{oh} - r^h) \geq \inf_{h\in H}\{p^* \cdot y | y \in \Gamma^h \cup \{0\}\}$. Let H^+ denote the subset of H so that $p^* \cdot (x^{oh} - r^h) \geq 0$. Let H^- denote the subset of H so that $p^* \cdot (x^{oh} - r^h) < 0$.

It is useful here to establish an identity

$\sum_{h\in H^+} \inf\{p^* \cdot y | y \in \Gamma^i \cup \{0\}\} + \sum_{h\in H^-} \inf\{p^* \cdot y | y \in \Gamma^i \cup \{0\}\} = \inf\{p^* \cdot y | y \in \Omega\}$

$\sum_{h\in H^+} p^* \cdot (x^{oh} - r^h) \geq \sum_{h\in H^+} \inf\{p^* \cdot y | y \in \Gamma^h \cup \{0\}\}$

$\geq \sum_{h\in H^+} \inf\{p^* \cdot y | y \in \Gamma^h \cup \{0\}\} + \sum_{h\in H^-} \inf\{p^* \cdot y | y \in \Gamma^h \cup \{0\}\}$

$= \sum_{h\in H} \inf\{p^* \cdot y | y \in \Gamma^h \cup \{0\}\}$

$= \inf\{p^* \cdot y | y \in \Omega\}$

$= \inf\{p^* \cdot y | y \in con(\Omega)\} \geq k$

$\geq \sup\{p^* \cdot y | y \in Z\} = -M$.

The core allocation x^{oh} is attainable, so $\sum_{h\in H}(x^{oh} - r^h) \leq 0$ and for any goods n in surplus at the core allocation $p^*_n = 0$. So $\sum_{h\in H} p^* \cdot (x^{oh} - r^h) = 0$. Then $\sum_{h\in H^-} p^* \cdot (x^{oh} - r^h) = -\sum_{h\in H^+} p^* \cdot (x^{oh} - r^h) \geq \inf\{p^* \cdot y | y \in \Omega\} \geq -M$

Now the conclusions of the theorem follow directly.

$\sum_{h\in H^-} |p^* \cdot (x^{oh} - r^h)| = \sum_{h\in H^+} |p^* \cdot (x^{oh} - r^h)| \leq M$, so

$\sum_{h\in H} |p^* \cdot (x^{oh} - r^h)| = \sum_{h\in H^-} |p^* \cdot (x^{oh} - r^h)| + \sum_{h\in H^+} |p^* \cdot (x^{oh} - r^h)| \leq 2M$. This establishes the assertion (i) in the Theorem.

To demonstrate assertion (ii) we form the following argument.

$\sum_{h\in H} |\inf\{p^* \cdot (x - r^h) | x \succ_h x^{oh}\}|$

$= \sum_{h\in H^+} |\inf\{p^* \cdot y | y \in \Gamma^h\}| + \sum_{h\in H^-} |\inf\{p^* \cdot y | y \in \Gamma^h\}|$

$\leq [-\sum_{h\in H^+} \inf\{p^* \cdot y | y \in \Gamma^h \cup \{0\}\} + \sum_{h\in H^+} p^* \cdot (x^{oh} - r^h)]$ - $\sum_{h\in H^-} \inf\{p^* \cdot y | y \in \Gamma^h \cup \{0\}\}$ (Wrestling with the algebra here will convince the reader that the term in square brackets is larger than the first term of the previous expression and — taking account of signs — the last term exceeds the last term of the previous expression).

$= -\sum_{h\in H^+} \inf\{p^* \cdot y | y \in \Gamma^h \cup \{0\}\} - \sum_{h\in H^-} \inf\{p^* \cdot y | y \in \Gamma^h \cup \{0\}\} + \sum_{h\in H^+} p^* \cdot (x^{oh} - r^h)$ (Then using the identity at the start of this step,)

$= -\inf\{p^* \cdot y | y \in \Omega\} + \sum_{h\in H^+} p^* \cdot (x^{oh} - r^h)$

$\leq M + M = 2M$.

QED