Part B

An economy with bounded production technology, supply and demand functions

In Chapters 4–7 we will develop a version of the complete Arrow-Debreu model of the economy. The theory of the firm and production sector is presented in Chapter 4 and that of households and demand in Chapter 5. We bring them together with the Walras' Law in Chapter 6 and existence of general equilibrium in Chapter 7.

As we noted in Chapter 2, the typical characterization of economic activity of firms and households is as a maximization subject to constraint. Recall the corollary to Theorem 2.7. In order for maximization to be well defined, sufficient conditions are that the maximand be a continuous function of its arguments and that the opportunity set be compact. That pretty well sets the agenda for characterizing firm and household behavior. We have to find continuous functions for them to maximize. We should find compact constraint sets for them to do it on. That will characterize firm supply and household demand behavior. Although these are not necessary conditions, they are the best generally sufficient conditions available.

Finding continuous functions for the firm and household to optimize does not pose a problem. For the firm, the obvious choice is profits. For the household the traditional maximand is utility, though we will go to some effort to derive the continuous utility function from the more primitive assumption of a preference ordering. The obvious constraint set for the firm is a representation of the firm's technically available possibilities–the possible input-output combinations based on available technology represented as a subset of $\mathbf{R}^N$. For the household, the obvious constraint set is a budget constraint. Are these constraint sets compact? If so we've satisfied the sufficient conditions for finding a well-defined maximum. Are the constraint sets closed and bounded?

Closedness is largely a technical concern and we don't really regard it as a problem. Boundedness is more difficult to establish. Is the firm's technology bounded? We will represent the firm's technology as a subset of $\mathbf{R}^N$. Any technically possible combination of inputs and outputs should be represented in the technology set. In a finite world with a finite economy, how could this set be unbounded? The answer is that in a finite world with a finite economy, realized outputs must be bounded in equilibrium. Firms and households should be led to these finite equilibrium outputs by prices. It should be a result in equilibrium, not an assumption at the outset of the study, that supplies offered to the market by firms are finite. The firm should be in a position to consider what it would produce if it could afford to buy arbitrarily large amounts of inputs. Eventually, equilibrium prices should persuade the firm that arbitrarily large production plans are unprofitable.

However, this leaves us with a difficult technical problem. How can we allow the firm to consider arbitrarily large (unbounded) production plans? If we allow the firm to try to optimize a profit function over an unbounded set (a noncompact set) we have no assurance that the firm's maximizing choice will be well defined. Without a well-defined maximum, we have no worthwhile theory of supply.

We face the same problem with the theory of the household. In equilibrium, the household will surely choose bounded consumption plans; after all, in a finite world bounded consumption is all that the economy will be able to produce so bounded demand will clear markets. That decision should however be the result of household optimization led by prices, not the outcome of exogenous constraint. Conversely, at disequilibrium prices, the household may face an unbounded budget set (if some goods have zero prices at the price vector currently proposed by the Walrasian auctioneer). If the budget set is not compact, how can we describe the demand behavior of the household? There may be no well-defined utility maximum if the constraint set is not compact.

The solution to this nest of difficulties is a rather elaborate two-step procedure. At first we will consider an economy with bounded production technology. The firms will maximize profits over their bounded technology sets. Attainable outputs will necessarily be bounded as well. Households will face bounded choice sets that are carefully constructed to include the attainable consumptions as a proper subset. We will demonstrate the existence of a general equilibrium in this economy with bounded firm technology and bounded individual choice sets. That comprises the agenda for this Part B.

The argument will then extend the model to the case of unbounded firm technologies. The resource endowment of the economy is, however, finite. Under reasonable weak assumptions we can show that the attainable outputs of the economy are finite. We then show that we can artificially restrict the unbounded taste and technology sets of this economy to a bounded subset containing the attainable set as a proper subset. This essentially reduces the problem of the economy with unbounded technology to the previous case of bounded technology. We will find an equilibrium in this artificially bounded economy. Then the rabbit comes out of the hat. We can show that the equilibrium of the artificially bounded economy is also an equilibrium of the original unbounded economy. That's the plan for Part C.

4

Production with bounded firm technology

4.1 Firms and production technology

We will represent production as organized in firms. A firm is characterized by its name, by the production technology to which it has access, and by who owns it, the shareholders. We postpone discussion of the ownership and distribution of profits to Chapter 6. The population of firms is the finite set F, indexed $j = 1, \ldots, \#F$. The typical firm is $j \in F$. Firm j's most distinctive characteristic is its production technology, represented by the nonempty set $\mathcal{Y}^j \subset \mathbf{R}^N$.

The set $\mathcal{Y}^j$ represents the technical possibilities of firm j. A typical element y of the technology set, $y \in \mathcal{Y}^j$, is a vector representing a technically possible combination of inputs and outputs. Negative coordinates of y are inputs; positive coordinates are outputs. For example, say $y \in \mathcal{Y}^j$, $y = (-2, -3, 0, 0, 1)$; this $y \in \mathcal{Y}^j$ means that an input of two units of good 1 and three units of good 2 will allow firm j to produce one unit of good 5. Each element y of $\mathcal{Y}^j$ is like a recipe in a cookbook or one of many blueprint plans for production, which can be implemented as a matter of choice by the firm. There is no guarantee that the economy can provide the inputs $y \in \mathcal{Y}^j$ specifies, either from endowment or from the output of other firms. Rather, $y \in \mathcal{Y}^j$ represents the technical output possibilities of production by firm j if the specified inputs are provided. A typical $\mathcal{Y}^j$ is illustrated in Figure 4.1. A point y in $\mathcal{Y}^j$ represents the answer to a hypothetical question: If the inputs specified in y were available what outputs could firm j produce? The answer includes the outputs (positive coordinates) specified in y.

The more common representation of a firm's production technology is a production function. How does a production function relate to a technology set $\mathcal{Y}^j$? The answer is that the production function embodies a concept of efficiency; the production function is the equation of the upper boundary

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Fig. 4.1. $\mathcal{Y}^j$: Technology set of firm j.

of $\mathcal{Y}^j$. In Figure 4.1, the curve depicting the implied production function is the line 0A. Think of firm j with the production function $w = f^j(x)$ where x is the (scalar) input to production and w is the scalar output. Let j's technology set be $\mathcal{Y}^j$ with a typical element $(-x, w)$. Then the relation between $f^j(x)$ and $\mathcal{Y}^j$ is $f^j(x) \equiv \max\{w \mid (-x, w) \in \mathcal{Y}^j\}$.

4.2 The form of production technology

We now formalize the analytic properties of $\mathcal{Y}^j$ as a subset of $\mathbf{R}^N$. We will use these assumptions to develop the theory of production and firm supply.[1]

(P.I) $\mathcal{Y}^j$ is convex for each $j \in F$.
(P.II) $0 \in \mathcal{Y}^j$ for each $j \in F$.
(P.III) $\mathcal{Y}^j$ is closed for each $j \in F$.

P.I is the convexity assumption. It corresponds to the idea of increasing marginal costs and diminishing marginal product. It says (when combined with P.II) that if a particular production plan is possible, then it is also possible at half the original scale. Hence P.I is an assumption that there are no scale economies and no indivisibilities.

P.II is the assumption that it is always possible to run a firm at a nil output level with nil inputs as well. That means that the worst the owners of the firm can do in terms of profits is zero. The firm is never required to operate at a loss. As a mathematical formality, this convention allows us to treat the formation of "new" firms in a quite general fashion as a special case of the ordinary analysis of firm production choices. At some prices the firm will find it unprofitable to produce; it will set output at zero and have zero profits. Prices may then change, making it attractive to produce at a positive output level instead of zero. This looks very much like the founding of a new firm, based on the renewed profitability of its line of work. In the formal statement of the model, the "new" firm has always been there, operating at a nil level.

P.III is essentially technical, helping to assure us of a well-defined profit

[1] We will designate assumptions on the structure of production by "P." and those on the structure of consumption by "C." followed by a roman numeral. The numbering of the assumptions will differ from their order of appearance (resulting in consecutive low-numbered assumptions in the most general model, Chapter 17).

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Fig. 4.2. Convex and nonconvex technology sets.

maximizing production plan for the firm and of the continuity of output decisions with prices.

We will introduce P.IV and P.V later. Here we will skip to P.VI:

(P.VI) $\mathcal{Y}^j$ is a bounded set for each $j \in F$.

P.VI is a very convenient assumption; it is also very restrictive. The convenience comes from our notions of how to describe firm behavior–profit maximization. P.III says that $\mathcal{Y}^j$ is closed and now P.VI says that it is bounded. Hence, under P.III and P.VI, $\mathcal{Y}^j$ is a compact set. Maximizing profits over this domain should result in a well-defined answer (Corollary 2.2).

4.3 Strictly convex production technology

We wish to describe firm supply behavior as profit maximization subject to technology constraint. In order to discuss the simplest possible case of firm supply behavior we introduce:

(P.V) For each $j \in F$, $\mathcal{Y}^j$ is strictly convex.

The assumption of strict convexity assures us of a unique (point-valued) profit-maximizing choice of production plan. Supply will be a function rather than set-valued (Theorem 4.1 below). This is very convenient and significantly simplifies the exposition and mathematics used. Note that P.V implies P.I; thus it is redundant to assume both. We can generalize to the case of weak convexity and set-valued supply behavior at some increase in technical detail. This exercise is performed in Part G (Chapters 16 and 17). Figure 4.2 illustrates three possible forms of $\mathcal{Y}^j$: strictly convex[2] (consistent with P.I and P.V), weakly convex (consistent with P.I but not P.V), and nonconvex (inconsistent with both).

We are now ready to develop a supply function for firm j. We start with a space of possible price vectors. We will describe prices by a vector $p \in \mathbf{R}^N_+$, $p = (p_1, p_2, \ldots, p_N)$, $p \neq 0$, where 0 denotes the zero vector in $\mathbf{R}^N$. $\mathbf{R}^N_+$ denotes the nonnegative orthant (quadrant) of $\mathbf{R}^N$. Thus the price vector is

[2] Since profit-maximizing choices will typically occur at the origin or above the horizontal axis, the figure illustrates the technology sets only in this region. In Figure 4.2a, please use your imagination to fill in the set below the axis to maintain strict convexity.

taken to have no negative coordinates and some strictly positive coordinates. Taking price vector $p \in \mathbf{R}^N_+$ as given,[3] each firm j "chooses" $y^j \in \mathcal{Y}^j$ such that $p \cdot y^j$ maximizes $p \cdot y$, the profits of the firm at production plan y, subject to inclusion in $\mathcal{Y}^j$. The sign convention, that inputs are negative co-ordinates of y and outputs are positive, means that $p \cdot y$ is the sum of the value of outputs minus the sum of the value of inputs, revenue minus cost equals profit. We define the supply function[4] of firm j as

$$\tilde{S}^j(p) = \{y^{*j} \mid y^{*j} \in \mathcal{Y}^j, p \cdot y^{*j} \geq p \cdot y \text{ for all } y \in \mathcal{Y}^j\}.$$

Then we have :

Theorem 4.1 Assume P.II, P.III, P.V, and P.VI. Let $p \in \mathbf{R}^N_+, p \neq 0$. Then $\tilde{S}^j(p)$ is well-defined, nonempty, and point-valued (a function). $\tilde{S}^j$ is continuous on $p \in \mathbf{R}^N_+, p \neq 0$.

Proof Well defined: $\tilde{S}^j(p)$ consists of the maximizer of a continuous function on a compact, nonempty, strictly convex set. The function is well defined since a continuous real-valued function achieves its maximum on a compact set (by Corollary 2.2).

Point-valued: We will demonstrate that the strict convexity of $\mathcal{Y}^j$ (P.V) implies that $\tilde{S}^j(p)$ is point-valued. We wish to show that there is a unique $y^0 \in \mathcal{Y}^j$ that maximizes $p \cdot y$ in $\mathcal{Y}^j$. Suppose not. Then there are $y^1, y^2 \in \mathcal{Y}^j$, $y^1 \neq y^2$, so that $p \cdot y^1 = \max_{y \in \mathcal{Y}^j} p \cdot y = p \cdot y^2$. Now consider the profitability of a convex combination of y^1 and y^2. For $0 < \alpha < 1$, $p \cdot [\alpha y^1 + (1-\alpha)y^2] = p \cdot y^1 = p \cdot y^2$. But, by strict convexity of $\mathcal{Y}^j$ (P.V), $[\alpha y^1 + (1-\alpha)y^2] \in$ interior $\mathcal{Y}^j$. That means that in a neighborhood of $[\alpha y^1 + (1-\alpha)y^2]$ there is $y^3 \in \mathcal{Y}^j$ so that $p \cdot y^3 > p \cdot y^1 = p \cdot y^2$, which is a contradiction. Hence, we conclude that $\tilde{S}^j(p)$ is point-valued and we can now validly represent $\tilde{S}^j(p)$ as a function.

Continuity: We now wish to demonstrate continuity of $\tilde{S}^j(p)$. Let $p^\nu \in \mathbf{R}^N_+, \nu = 1, 2, \ldots, p^\nu \neq 0, p^\nu \to p^0 \neq 0$. We must show that $\tilde{S}^j(p^\nu) \to \tilde{S}^j(p^0)$. Because $\tilde{S}^j(p^\nu)$ is a sequence in the compact set $\mathcal{Y}^j$, it contains a convergent subsequence. It is sufficient to show that the subsequence converges to $\tilde{S}^j(p^0)$; this will demonstrate that all subsequences converge to $\tilde{S}^j(p^0)$ and hence that $\tilde{S}^j$ is continuous.

Without loss of generality let $\tilde{S}^j(p^\nu) \to y^*$. We must show that $y^* = \tilde{S}^j(p^0)$. Suppose not. Then $p^0 \cdot \tilde{S}^j(p^0) > p^0 \cdot y^*$. But the dot product is

[3] Nonnegativity of prices reflects monotone preferences (desirability of goods), a concept to be introduced in the next chapter.

[4] The superscript tilde (˜) notation emphasizes that the supply function is defined over the bounded domain $\mathcal{Y}^j$.

a continuous function of its arguments, so for ν large, this implies that $p^\nu \cdot \tilde{S}^j(p^0) > p^\nu \cdot \tilde{S}^j(p^\nu)$, which is a contradiction. Hence $\tilde{S}^j(p^\nu) \to \tilde{S}^j(p^0)$.

This completes the proof. QED

Lemma 4.1 (homogeneity of degree 0) Assume P.II, P.III, P.V, and P.VI. Let $\lambda > 0, p \in \mathbf{R}^N_+, p \neq 0$. Then $\tilde{S}^j(\lambda p) = \tilde{S}^j(p)$.

4.3.1 Aggregate Supply

We now wish to move from the behavior of the individual firm to production plans of the whole productive sector. The definition of individual firm j's technology as $\mathcal{Y}^j \subseteq \mathbf{R}^N$ is stated without reference to other firms' production plans. This expresses the notion that there are no external effects in production — firm j's production decisions can be made independent of other firms' choices. Supply behavior for the economy as a whole is the summation over all firms $j \in F$ of their individual supply functions $\tilde{S}^j(p)$. That is

Definition For any $p \in \mathbf{R}^N_+, p \neq 0$, the economy's aggregate supply function is $\tilde{S}(p) \equiv \sum_{j \in F} \tilde{S}^j(p)$.

This definition leads to

Theorem 4.2 Assume P.II, P.III, P.V, and P.VI. Let $p \in \mathbf{R}^N_+, p \neq 0$. Then $\tilde{S}(p)$ is well-defined, nonempty, and point-valued (a function). $\tilde{S}$ is continuous on $p \in \mathbf{R}^N_+, p \neq 0$.

Proof: Theorem 4.1.

4.4 Attainable production plans

Recall

Definition A sum of sets $\mathcal{Y}^j$ in $\mathbf{R}^N$ is defined as

$$\mathcal{Y} = \sum_j \mathcal{Y}^j \text{ is the set } \left\{ y \mid y = \sum_j y^j \text{ for some } y^j \in \mathcal{Y}^j \right\}.$$

We will now define the economy's aggregate technology set as $\mathcal{Y} \equiv \sum_{j \in F} \mathcal{Y}^j$. The definition $\mathcal{Y} = \sum_j \mathcal{Y}^j$ again emphasizes independence, that there are no external effects in production. Production decisions of the individual firms can be combined additively. Note that in some co-ordinates some firms will have negative values (inputs) and other firms will have positive values (outputs). These intermediate goods are netted out in the summation. What is left is $y \in \mathcal{Y}$ whose negative co-ordinates are net inputs to the economy's production plans and positive co-ordinates are net outputs. We are interested in the array of outputs that can be achieved by the economy. The economy's initial endowment of resources is denoted $r \in \mathbf{R}^N_+$. r is supplied to the economy, to household consumers and to firms in the production sector.

Definition Let $y \in \mathcal{Y}$. Then y is said to be ***attainable*** if $y + r \geq 0$ (the inequality applies co-ordinatewise).

That is, a production plan is attainable if the economy's initial resources are sufficient to provide its net input requirements. Note that attainability is defined for $\mathcal{Y}$, the aggregate technology, not for $\mathcal{Y}^j$, the individual firm technologies.

Restating the definition, $y \in \mathcal{Y}$ is attainable if $x = y + r$, where $x \in [\mathcal{Y} + \{r\}] \cap \mathbf{R}^N_+$. This definition takes the aggregate production technology set $\mathcal{Y}$, translates it by the endowment vector r, and then takes the intersection with the nonnegative orthant (quadrant) of $\mathbf{R}^N$, $\mathbf{R}^N_+$. The intersection is the set of x attainable as outputs or aggregate consumptions (attainable production plans plus endowment). This intersection corresponds to the 90° wedge-shaped attainable set bounded by the coordinate axes and the production frontier in the Robinson Crusoe model, the set (designating it by points on its boundary) 0ABCDMSHGFE in Figures 1.1 and 1.2.

Since the attainable production vectors are those that can be produced with the available resources (and hence do not create unsatisfiable excess demands in factor markets), it is among these that an equilibrium vector is to be found (if it exists). Since $\mathcal{Y}^j$ is bounded by P.VI, $\mathcal{Y}$ (as the finite sum of $\mathcal{Y}^j$) is bounded and therefore trivially, so is the attainable subset of $\mathcal{Y}$.

4.5 Bibliographic note

The presentation of production technology in this chapter parallels that of Arrow (1962) and Arrow and Debreu (1954). It is simplified here–the full Arrow-Debreu treatment appears in Chapters 8 and 17.

Exercises

4.1 Consider the model of the firm and production presented in Chapter 4. Theorem 4.1 (or parts of it) is false if we omit P.VI, boundedness of $\mathcal{Y}^j$.

(i) Explain mathematically how and why Theorem 4.1 fails.

(ii) Demonstrate by example that Theorem 4.1 fails. Explain the example.

4.2 Consider the following production function representing the technology of one firm. Production of y involves a set-up cost, $S > 0$, which is the initial amount of input required before any positive production can take place. We have

$$y = \begin{cases} 0 & if L \leq S \\ [3pt] a(L - S) & if L > S, \end{cases}$$

where L is the amount of labor used as an input to y, and a is a positive constant. This production function (like any production function) is the upper boundary of a technology set.

Show that this production function or its technology set violates the (weak) convexity assumption (P. I). Discuss.

4.3 In the Robinson Crusoe model of Chapter 1, we implicitly used the assumption of convex technology, describing the production possibility set as convex. Consider a Robinson Crusoe economy with a nonconvex production possibility set.

(i) Diagram the possibility that there is a competitive equilibrium (despite the nonconvexity).

(ii) Is the equilibrium established in (i) Pareto efficient? Explain.

(iii) Diagram the possibility that there is no competitive equilibrium (due to the nonconvexity). Explain.

(iv) In the nonconvex Robinson Crusoe economy, can a Pareto efficient allocation generally be sustained as a competitive equilibrium? Diagram and explain.