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General equilibrium of the market economy with an excess demand correspondence

17.1 General equilibrium with set-valued supply and demand

Our plan in this chapter is to take the model of production, consumption, the economy, and market equilibrium of Chapters 8–11[1] and restate it for the case of set-valued demand and supply behavior. Formally this means that we dispense with assumptions of strict convexity of tastes and production technology, C.VII and P.V. We rely rather on convexity, C.VI and P.I. Under the remaining assumptions on consumption and production behavior, this will allow us to characterize demand and supply behavior as upper hemicontinuous, convex-valued correspondences. In turn, excess demand will then be characterized as upper hemicontinuous and convex valued. A model of price adjustment that is also upper hemicontinuous and convex valued completes the picture: Applying the Kakutani Fixed-Point Theorem allows us to find a fixed point in price space that achieves a market equilibrium.

Just as we did in Chapters 8–11, we treat the economy in two formats: an artificially restricted bounded economy denoted by the superscript tilde notation ($\sim$) and an unrestricted economy (representing the true model we are really interested in). The artificially restricted economy is a purely technical construct, designed to allow us to develop the properties of the underlying unrestricted economy in a more tractable setting. The technique of the proof is to note that the restricted budget, demand, supply, and profit behavior is always well defined, since it represents optimizing behavior on a compact set. Unrestricted demand and supply correspondences and profit functions may not be everywhere well defined. When the demand and supply correspondences of the restricted economy designate attainable allocations, then they coincide with their counterparts of the unrestricted economy. An

[1] Note that the model of these chapters includes as a special case the bounded economy model of Chapters 4–7.

equilibrium allocation is necessarily attainable. Hence when we find an equilibrium of the artificially restricted economy (something that is possible for us to do since its behavior is everywhere upper hemicontinuous, convex valued, and well defined), the equilibrium price vector and allocation is also an equilibrium of the unrestricted (true) economy.

17.2 Production with a (weakly) convex production technology

We will show that supply behavior of the firm is convex and set valued when the production technology is convex but not strictly convex. This includes the cases of constant returns to scale, linear production technology, and perfect substitutes among inputs to production. In each of these cases there may be a (linear) range of equally profitable production plans differing by scale of output or by the input mix. The purpose of developing a theory of set-valued supply behavior is to accommodate this range of indeterminacy.

Supply correspondence with a weakly convex production technology: We now omit P.V and use P.I–P.IV only. In this case the policy of profit maximization for firm j may not yield a unique solution.

Let $S^j(p) = \{y^*|y^* \in Y^j, p \cdot y^* \geq p \cdot y \text{ for all } y \in Y^j\}$ be the supply correspondence of the firm.

Example 17.1 An upper hemicontinuous, convex-valued supply correspondence. Let firm j's production technology be described as follows.

Let $Y^j = \{(x, y)|y \leq -x; x \leq 0, K \geq y \geq 0\}$. That is, output y is produced by a constant returns technology using input x, each unit of x producing one unit of y, up to a limit of K of y. Let the price vector p be an element of the price space $\mathbf{R}^2_{++} = \{(p_x, p_y)|p_x, p_y > 0\}$. Then for each $p \in \mathbf{R}^2_{++}$, we have the supply correspondence

$$\begin{aligned} S^j(p) &= \{(0,0)\} \text{ for } p_x > p_y, \\ &= \{(-y, y)|y \in [0, K]\} \text{ for } p_x = p_y, \\ &= \{(-K, K)\} \text{ for } p_x < p_y. \end{aligned}$$

Note that starting with the convex technology set Y^j, the resulting supply correspondence $S^j(p)$ is also convex valued. The correspondence is upper hemicontinuous (it has a closed graph). $S^j(p)$ is depicted in Figure 17.1. Note that with upper hemicontinuity and convex valuedness, a continuous downward-sloping demand curve will intersect the supply correspondence. The importance of the convexity of Y^j is demonstrated by comparison to Example 17.2, below.

Fig-17.1.eps

Fig. 17.1. Example 17.1–An upper hemicontinuous, convex-valued supply correspondence.

Fig-17.2.eps

Fig. 17.2. Example 17.2–an upper hemicontinuous supply correspondence that is not convex-valued.

Example 17.2 An upper hemicontinuous supply correspondence that is not convex valued. We consider here the supply behavior of a firm situated similarly to Example 17.1 with a minimum efficient scale of output,

$$Y^j = \{(x, y) | y \leq -x; K \geq y \geq 0 \text{ for } x \leq -k; y = 0 \text{ for } 0 \geq x > -k\}$$

Y^j is a nonconvex set, representing the scale economy. Minimum efficient scale of output is k; inputs insufficient to support output of k result in a zero output. This technology set gives us a supply correspondence that is upper hemicontinuous, but not convex valued:

$$\begin{aligned} S^j(p) &= \{(0, 0)\} \text{ for } p_x > p_y, \\ &= \{(-y, y) | y = 0 \text{ or } y \in [k, K]\} \text{ for } p_x = p_y, \\ &= \{(-K, K)\} \text{ for } p_x < p_y. \end{aligned}$$

$S^j(p)$ is depicted in Figure 17.2. Note the jump in the supply correspondence at $p_x = p_y$. This jump is sometimes loosely described as a discontinuity. That description is imprecise, since the correspondence is actually upper hemicontinuous. Rather, the correspondence is nonconvex-valued at $p_x = p_y$. The example demonstrates the importance of convex valuedness for the existence of market equilibrium. A continuous downward-sloping demand curve may have no intersection with $S^j(p)$, hence implying no market equilibrium. Upper hemicontinuity of demand and supply is insufficient to assure a market equilibrium. Convex valuedness of the correspondence may be needed as well.

Taking price vector $p \in \mathbf{R}^N_+$ as given, each firm j "chooses" y^j in Y^j. Profit maximization guides the choice of y^j. Firm j chooses y^j to maximize $p \cdot y$ subject to $y \in Y^j$. We will consider two cases:

- a restricted supply correspondence where the supply behavior of firm j is required to be in a compact convex set $\tilde{Y}^j \subseteq Y^j$, which includes the plans attainable in Y^j as a proper subset, and

- an unrestricted supply correspondence where the only requirement is that the chosen supply behavior lie in Y^j. Of course, Y^j need not be compact. Hence, in this case, profit-maximizing supply behavior may not be well defined. Further, Y^j may include unattainable production plans. When the profit-maximizing production plan is unattainable, it cannot, of course, be fulfilled and cannot represent a market equilibrium.

The restricted supply correspondence will be denoted $\tilde{S}^j(p) \subset \tilde{Y}^j$, and the unrestricted supply correspondence will be $S^j(p) \subset Y^j$.

Recall Theorems 8.1 and 8.2. They demonstrated that under assumptions P.I, P.II, P.III, and P.IV the set of attainable production plans for the economy and for firm j were bounded. We then defined $\tilde{Y}^j$ as the bounded subset of Y^j containing production plans of Euclidean length c or less, where c was chosen as a strict upper bound on all attainable plans in Y^j. That is, choose c such that $|y^j| < c$ (a strict inequality) for y^j attainable in Y^j. Let $\tilde{Y}^j = Y^j \cap \{y||y| \leq c\}$. Note the weak inequality in the definition of $\tilde{Y}^j$. Restricting attention to $\tilde{Y}^j$ in describing firm j's production plans allows us to remain in a bounded set so that profit maximization will be well defined. Note that $\tilde{Y}^j$ is nonempty, closed, bounded (hence compact), and convex.

Define the restricted supply correspondence of firm j as

$$\tilde{S}^j(p) = \{y^{*j}|p \cdot y^{*j} \geq p \cdot y^j \text{ for all } y^j \in \tilde{Y}^j, y^{*j} \in \tilde{Y}^j\}.$$

In many of the lemmas and theorems below assumptions P.I - P.IV are introduced because the restriction to $\tilde{Y}^j$ is essential to the analysis and this restriction rests on the boundedness of production plans attainable in Y^j.

The (unrestricted) supply correspondence of firm j was defined above as

$$S^j(p) = \{y^*|y^* \in Y^j, p \cdot y^* \geq p \cdot y \text{ for all } y \in Y^j\}.$$

Then we have:

Lemma 17.1 Under P.I–P.IV, $\tilde{S}^j(p)$ is convex.

Proof Let $y^1 \in \tilde{S}^j(p)$ and $y^2 \in \tilde{S}^j(p)$. For fixed $p, p \cdot y^1 = p \cdot y^2 \geq p \cdot y$ for all $y \in Y^j$. For $0 \leq \lambda \leq 1$, consider

$$p \cdot [\lambda y^1 + (1-\lambda)y^2] = \lambda p \cdot y^1 + (1-\lambda)p \cdot y^2 = p \cdot y^2 \geq p \cdot y$$

for all $y \in Y^j$.

But $(\lambda y^1 + (1-\lambda)y^2) \in Y^j$ by P.I. QED

Lemma 17.2 Under P.I–P.IV, $\tilde{S}^j(p)$ is nonempty and upper hemicontinuous for all $p \in \mathbf{R}^N_+, p \neq 0$.

Proof The set $\tilde{S}^j(p)$ consists of the maximizers of a continuous real-valued function on a compact set. The maximum is hence well defined and the set is nonempty.

To demonstrate upper hemicontinuity, let $p^\nu \to p^0; p^\nu, p^0 \in \mathbf{R}_+^N; p^\nu, p^0 \neq 0; \nu = 1, 2, \ldots;$ and $y^\nu \in \tilde{S}^j(p^\nu), y^\nu \to y^0$.
We must show that $y^0 \in \tilde{S}^j(p^0)$. Suppose not. Then there is $y' \in \tilde{Y}^j$ so that $p^0 \cdot y' > p^0 \cdot y^0$. The dot product is a continuous function:

$$p^\nu \cdot y' \to p^0 \cdot y'$$
$$p^\nu \cdot y^\nu \to p^0 \cdot y^0.$$

Therefore, for ν sufficiently large, $p^\nu \cdot y' > p^\nu \cdot y^\nu$. But this contradicts the definition of $\tilde{S}(p^\nu)$. The contradiction proves the lemma. QED

Theorem 17.1 Assume P.I - P.IV. Then

(a) $\tilde{S}^j(p)$ is an upper hemicontinuous correspondence throughout P. For each $p, \tilde{S}^j(p)$ is closed, convex, bounded, and nonnull;
(b) $\tilde{\pi}^j(p)$ is a well-defined continuous function for all $p \in \mathbf{P}$;
(c) if y^j is attainable in Y^j and $y^j \in \tilde{S}^j(p)$, then $y^j \in S^j(p)$.

Proof Part (a). Upper hemicontinuity and nonemptiness are established in Lemma 17.2. $\tilde{S}^j(p)$ is bounded since $\tilde{Y}^j$ is bounded. Closedness follows from upper hemicontinuity. Convexity is established in Lemma 17.1.

Part (b): For each $p \in \mathbf{P}$, $\tilde{S}^j(p)$ is nonempty and for any two $y', y'' \in \tilde{S}^j(p)$, $p \cdot y' = p \cdot y'' = \tilde{\pi}^j(p)$. Let $p^\nu \in \mathbf{P}$, $\nu = 1, 2, ..., p^\nu \to p^o$. Let $y^\nu \in \tilde{S}^j(p^\nu)$. Without loss of generality — since $\tilde{Y}^j$ is compact — let $y^\nu \to y^o$. The dot product is a continuous function of its arguments so $\tilde{\pi}^j(p^\nu) = p^\nu \cdot y^\nu \to p^o \cdot y^o = \tilde{\pi}^j(p^0)$. Thus $\tilde{\pi}^j(p)$ is continuous throughout P.

Part (c): Suppose y^j attainable and $y^j \in \tilde{S}^j(p)$ but $y^j \notin S^j(p)$. Then there is $\hat{y}^j \in Y^j$ so that $p \cdot \hat{y}^j > p \cdot y^j$. Furthermore,

$$p \cdot [\alpha \hat{y}^j + (1 - \alpha) y^j] > p \cdot y^j \text{ for any } \alpha, 0 < \alpha \leq 1.$$

But for α sufficiently small,

$$|\alpha \hat{y}^j + (1 - \alpha) y^j| \leq c,$$

so that

$$\alpha \hat{y}^j + (1 - \alpha) y^j \in \tilde{Y}^j.$$

But then $p \cdot (\alpha \hat{y}^j + (1 - \alpha) y^j) > p \cdot y^j$ and $\alpha \hat{y}^j + (1 - \alpha) y^j \in \tilde{Y}^j$; thus y^j is not the maximizer of $p \cdot y$ in $\tilde{Y}^j$ and $y^j \notin \tilde{S}^j(p)$ as was assumed. The contradiction proves the theorem. QED

Lemma 17.3 (homogeneity of degree 0) Assume P.I–P.IV. Let $\lambda > 0, p \in \mathbf{R}_{+}^{N}$. Then $\tilde{S}^j(\lambda p) = \tilde{S}^j(p)$ and $S^j(\lambda p) = S^j(p)$.

Proof Exercise 17.1.

17.3 Households

We now develop a theory of the household with set-valued demand behavior paralleling the theory of the household developed in Chapter 9. We use all of the structure and assumptions developed there with the exception of the assumption of strict convexity of preferences, C.VII. We use convexity, C.VI, which admits the possibility of set-valued linear segments in demand behavior, occurring, for example, in the case of perfect substitutes in consumption. To see how this might arise, consider Example 17.3.

Example 17.3 Convex set-valued household demand. Let household i's possible consumption set X^i be $\mathbf{R}_{+}^{2}$, the nonnegative quadrant in $\mathbf{R}^2$. Let the household endowment be (1, 1) with no ownership of shares of firms. At prices $p \in \mathbf{R}_{+}^{2}$, the household income is $p \cdot (1,1) = p_x + p_y$. Let household preferences be described by the utility function $u(x, y) = [ax + by]$. Then household demand can be characterized as

$$D^i(p) = \begin{cases} ([p_x + p_y]/p_x, 0) \text{ for } \frac{p_x}{p_y} < \frac{a}{b} \\ (0, [p_x + p_y]/p_y) \text{ for } \frac{p_x}{p_y} > \frac{a}{b} \\ \{(x, [p_x + p_y - p_x x]/p_y) | x \in [0, (p_x + p_y)/p_x]\} \text{ for } \frac{p_x}{p_y} = \frac{a}{b} \\ \text{undefined for } p_x = 0 \text{ or } p_y = 0. \end{cases}$$

Note that $D^i(p)$ is convex set valued for $p_x/p_y = a/b$. This simply reflects the idea that if goods x and y are perfect substitutes at the ratio a/b then, when their prices occur in this ratio, the household will be indifferent among a whole set of linear combinations of x and y in the inverse of this ratio. After all, if the goods x and y are perfect substitutes then it really doesn't matter in what proportion they are used. The demand behavior, $D^i(p)$, is described as upper hemicontinuous and convex valued for all p so that $p_x \neq 0$ and $p_y \neq 0$.

We now define the household's budget set and demand correspondences. The household budget set is precisely as defined in Chapter 9:

$$B^i(p) \equiv \{x | x \in \mathbf{R}^N, p \cdot x \leq M^i(p)\}.$$

The definition of demand behavior for household i is here just as it was in Chapter 9, but since we are using C.VI (convexity of preferences) rather than C.VII (strict convexity of preferences) we will be dealing with a demand correspondence rather than a demand function. We have

$$D^i \,:\, \mathbf{R}^N_+ \rightarrow \mathbf{R}^N,$$
$$D^i(p) \equiv \{y | y \in B^i(p) \cap X^i, y \succeq_i x \text{ for all } x \in B^i(p) \cap X^i\}$$
$$\equiv \{y | y \in B^i(p) \cap X^i, u^i(y) \geq u^i(x) \text{ for all } x \in B^i(p) \cap X^i\}.$$

We now define the artificially bounded budget and demand sets much as we did in Chapter 9. Choose c so that $|x|<c$ (a strict inequality) for all attainable consumptions x. Theorem 8.1 assures us that c exists under P.I - P.IV. The artificially restricted budget set is then defined as

$$\tilde{B}^i(p) = \{x | x \in \mathbf{R}^N, p \cdot x \leq \tilde{M}^i(p), |x| \leq c\}.$$

Note that $\tilde{B}^i(p)$ is just as defined in Chapters 5 and 9. $\tilde{B}^i(\cdot)$ is homogeneous of degree 0, just as is $B^i(\cdot)$. We now define the artificially restricted demand correspondence,

$$\tilde{D}^i(p) \equiv \{x | x \in \tilde{B}^i(p) \cap X^i, x \succeq_i y \text{ for all } y \in \tilde{B}^i(p) \cap X^i\}.$$

Note that $\tilde{D}^i(p)$ is just as defined in Chapters 5 and 9, but under weak convexity (C.VI) $\tilde{D}^i(p)$ may be set valued.

Just as in Chapters 6 and 10, firm j's profit function is $\pi^j(p) = \max_{y \in Y^j} p \cdot y$. Since Y^j need not be compact, $\pi^j(p)$ may not be well defined. Firm j's profit function in the artificially restricted firm technology set $\tilde{Y}^j$ is $\tilde{\pi}^j(p) = \max_{y \in \tilde{Y}^j} p \cdot y$. The function $\tilde{\pi}^j(p)$ is always well defined, since $\tilde{Y}^j$ is compact by definition and P.III.

Just as in Chapters 6 and 10, household i's income is defined as

$$M^i(p) = p \cdot r^i + \sum_{j \in F} \alpha^{ij} \pi^j(p).$$

For the model with restricted firm supply behavior, household income is

$$\tilde{M}^i(p) = p \cdot r^i + \sum_{j \in F} \alpha^{ij} \tilde{\pi}^j(p).$$

Note that $M^i(p)$ may not be everywhere well defined since $\pi^j(p)$ is not well defined for some j, p. Conversely, $\tilde{M}^i(p)$ is continuous, real valued, nonnegative, and well defined for all $p \in \mathbf{R}^N_+$. By the same argument as in Chapters 5 and 9, $\tilde{B}^i(p)$ and $\tilde{D}^i(p)$ are homogeneous of degree 0 in p. This allows us to confine attention in prices to the unit simplex in $\mathbf{R}^N$, denoted P.

As in Chapter 9, in order to avoid discontinuities in demand behavior at the boundary of X^i we will continue to assume C.VIII, positivity and sufficiency of income,

$$\tilde{M}^i(p) \gg \min_{x \in X^i \cap \{y | y \in \mathbf{R}^N, c \geq |y|\}} p \cdot x \geq 0 \text{ for all } p \in P.$$

We want to show that the (artificially restricted) demand correspondence of household i, $\tilde{D}^i(p)$, is upper hemicontinuous and convex valued. To demonstrate upper hemicontinuity, we will use the Theorem of the Maximum, Theorem 16.3. That theorem requires that the opportunity set, in this case $\tilde{B}^i(p) \cap X^i$, be continuous, both upper and lower hemicontinuous. Continuity of $\tilde{B}^i(p) \cap X^i$ is the message of Theorem 17.2.

Theorem 17.2 Assume P.I - P.IV, C.I, C.III, and C.VIII. Then $\tilde{B}^i(p) \cap X^i$ is continuous (lower and upper hemicontinuous), compact valued, and nonnull for all $p \in P$.

Proof P.I - P.IV and Theorem 8.1 ensure that c is well-defined. Continuity of $\tilde{B}^i(p) \cap X^i$ depends on continuity of $\tilde{M}^i(p)$. This follows from definition and Theorem 17.1 (continuity of $\tilde{\pi}^j(p)$). Upper hemicontinuity of $\tilde{B}^i(p) \cap X^i$ is left as an exercise. Nonnullness follows directly from C.VIII. Compactness follows from closedness and the restriction to $\{x | |x| \leq c\}$. To demonstrate lower hemicontinuity, we will use positivity of income, C.VIII, and the convexity of $\tilde{B}^i(p) \cap X^i$. Consider a sequence $p^\nu \in P, p^\nu \to p^0, y^0 \in \tilde{B}^i(p^0) \cap X^i$. To establish lower hemicontinuity we need to show that there is a sequence y^ν, so that $y^\nu \in \tilde{B}^i(p^\nu) \cap X^i$ and $y^\nu \to y^0$. We will consider two cases depending on the cost of y^0 at price vector p^0.

CASE 1 $p^0 \cdot y^0 > 0$ and

$$p^0 \cdot y^0 > \min_{x \in X^i \cap \{y | y \in \mathbf{R}^N, c \geq |y|\}} p^0 \cdot x.$$

The strategy of proof in this case is to create the required sequence y^ν in the following way. Find a minimum expenditure point, x^0 in $X^i \cap \{x | |x| \leq c\}$. We extend a ray from x^0 through y^0. We then take a sequence of points on the ray chosen to fulfill the budget constraint at p^ν and to converge to y^0. That sequence is y^ν. This construction is depicted in Figure 17.3.

Fig-17.3.eps

Fig. 17.3. Theorem 17.2–continuity of the budget set showing the construction of y^ν.

For ν large, we have

$$p^\nu \cdot y^0 > \min_{x \in X^i \cap \{y | y \in \mathbf{R}^N, c \geq |y|\}} p^0 \cdot x.$$

We choose x^0 as a cost-minimizing element of $X^i \cap \{x||x| \leq c\}$ at prices p^0. Let $x^0 \in X^i \cap \{x||x| \leq c\}$ and

$$p^0 \cdot x^0 = \min_{x \in X^i \cap \{y | y \in \mathbf{R}^N, c \geq |y|\}} p^0 \cdot x.$$

We now construct y^ν as a convex combination of x^0 and y^0, fulfilling budget constraint at p^ν.

$$\text{Let } \alpha^\nu = \min \left[1, \frac{[\tilde{M}^i(p^\nu) - p^\nu \cdot x^0]}{p^\nu \cdot (y^0 - x^0)} \right],$$
$$y^\nu = \alpha^\nu y^0 + (1 - \alpha^\nu) x^0.$$

For ν large, α^ν is well defined. y^ν is chosen here so that it fulfills budget constraint and converges to y^0. We have $p^\nu \cdot y^\nu = p^\nu \cdot ((1-\alpha^\nu)x^0 + \alpha^\nu y^0) \leq \tilde{M}^i(p^\nu)$. $\alpha^\nu \to 1$ as ν becomes large. By convexity of X^i (C.III), $y^\nu \in X^i \cap \{x||x| \leq c\}$. For ν large, $p^\nu \cdot x^0 < p^\nu \cdot y^0$ and $p \cdot y^\nu \leq \tilde{M}^i(p^\nu)$. So $y^\nu \in \tilde{B}^i(p^\nu) \cap X^i$ and $y^\nu \to y^0$. Hence the sequence y^ν demonstrates lower hemicontinuity of $\tilde{B}^i(p) \cap X^i$.

CASE 2 $p^0 \cdot y^0 = 0 < \tilde{M}^i(p^0)$ or

$$p^0 \cdot y^0 = \min_{x \in X^i \cap \{y | y \in \mathbf{R}^N, c \geq |y|\}} p^0 \cdot x.$$

Once again we need to construct a sequence y^ν with the required convergence properties. In this case it is trivial. By continuity of the dot product, for large ν, $p^\nu \cdot y^0 < \tilde{M}^i(p^\nu)$. By hypothesis we have $y^0 \in \tilde{B}^i(p^0) \cap X^i$. Thus we can set $y^\nu = y^0$; then for ν large, we have $y^\nu \in \tilde{B}^i(p^\nu) \cap X^i$ and hence $y^\nu \to y^0$ trivially.

Cases 1 and 2 exhaust the possibilities. In each case we have demonstrated the presence of sequence y^ν, so that $y^\nu \in \tilde{B}^i(p^\nu) \cap X^i$ and $y^\nu \to y^0$. This is precisely what lower hemicontinuity of $\tilde{B}^i(p) \cap X^i$ requires.

QED

Theorem 17.2 demonstrates the continuity of the consumer's opportunity set $\tilde{B}^i(p) \cap X^i$ as a function of p. We are not really interested in $\tilde{B}^i(p) \cap X^i$ on its own. Rather, we are interested in the household demand behavior, $\tilde{D}^i(p)$. In order to apply the Kakutani Fixed-Point Theorem and find a general equilibrium we would like $\tilde{D}^i(p)$ to be upper hemicontinuous and convex valued. Upper hemicontinuity follows from Theorem 17.2 and the Maximum Theorem (Theorem 16.3). This is demonstrated in Theorem 17.3.

Theorem 17.3 Assume P.I - P.IV, C.I, C.III, C.V, and C.VIII. Then $\tilde{D}^i(p)$ is an upper hemicontinuous nonnull correspondence for all $p \in P$.

Proof By Theorem 17.2 above, $\tilde{B}^i(p)$ is continuous with $\tilde{B}^i(p) \cap X^i$ nonempty, compact, continuous for all $p \in P$. By Theorem 5.1, $u^i(\cdot)$ is a continuous real-valued function. $\tilde{D}^i(p)$ is defined as the set of maximizers of $u^i(\cdot)$ on $\tilde{B}^i(p) \cap X^i$. Nonnullness follows since a continuous function achieves its maximum on a compact set. Upper hemicontinuity of $\tilde{D}^i(p)$ follows from the Maximum Theorem (Theorem 16.3). QED

Recall the convexity assumption

(C.VI) $x \succeq_i y$ implies $((1-\alpha)x + \alpha y) \succeq_i y,$ for $0 \ \leq \alpha \leq 1$.

Under C.VI, we have convexity of $\tilde{D}^i(p)$. This is formalized as Theorem 17.4.

Theorem 17.4 Assume P.I - P.IV, C.I, C.III, C.V, C.VI, and C.VIII. Then $\tilde{B}^i(p)$ and $\tilde{D}^i(p)$ are convex valued.

Proof Exercise 17.3.

Under weak monotonicity (C.IV) we can rely on households spending all of their available income subject to constraint. This is the implication of Lemmas 17.4 and 17.5 below.

Lemma 17.4 Under C.IV, C.V, and C.VI, $x \in D^i(p)$ implies $p \cdot x = M^i(p)$.

Proof Exercise 17.4.

Lemma 17.5 Under C.IV and C.V, $x \in \tilde{D}^i(p)$ implies $p \cdot x \leq \tilde{M}^i(p)$. Further, if $p \cdot x < \tilde{M}^i(p)$, then $|x| = c$.

Proof Exercise 17.5. The proof follows from weak monotonicity, C.IV. (See proof of Theorem 6.2.)

Lemma 17.6 Assume C.I - C.VI, $i \in H$, $x \in X^i, x' \in int(X^i), x' \succ_i x$, $0 < \alpha < 1$. Then $\alpha x' + (1-\alpha)x \succ_i x$.

Proof $x' \in int(X^i)$. Under C.II, C.IV, and C.V, there is $x'' \in X^i, x''_k < x'_k$, in all co-ordinates $k = 1, 2, ..., N$, so that $x \sim_i x''$. But C.VI implies $\alpha x + (1-\alpha)x'' \succeq_i x$ and C.IV implies $\alpha x + (1-\alpha)x' \succ_i \alpha x + (1-\alpha)x''$. So $\alpha x + (1-\alpha)x' \succ_i x$. QED

Lemma 17.7 Under P.I - P.IV, C.I–C.VI, and C.VIII, $\tilde{D}^i(p)$ is upper hemicontinuous, convex, nonnull, and compact for all $p \in P$. If $M^i(p)$ is well defined and $M^i(p) = \tilde{M}^i(p)$, if $x \in \tilde{D}^i(p)$ and x is attainable, then $x \in D^i(p)$.

Proof Upper hemicontinuity follows from Theorem 16.2. Convexity of $\tilde{D}^i(p)$ follows from convexity of preferences (C.VI) and convexity of $\tilde{B}^i(p)$(Theorem 17.4).

If $x \in \tilde{D}^i(p)$ and x is attainable then $|x| < c$. Note the strict inequality. We now wish to show that $x \in D^i(p)$. Suppose not. Then there is $x' \in B^i(p) \cap X^i$ so that $x' \succ_i x$. Under C.II, C.V, and C.VIII, x' may be chosen so that $x' \in int(X^i)$. But then by lemma 17.6, for all α, $0 < \alpha < 1$, $(1-\alpha)x + \alpha x' \succ_i x$. For α sufficiently small, then $(1-\alpha)x + \alpha x' \in \tilde{B}^i(p)$, but this is a contradiction since x is the optimizer of $\succeq_i$ in $\tilde{B}^i(p)$. QED

17.4 The market economy

We now bring the two sides, households and firms, of the set-valued economic model together. The demand correspondence of the unrestricted model is defined as

$$D(p) = \sum_{i \in H} D^i(p).$$

For the artificially restricted model, the demand side is characterized as

$$\tilde{D}(p) = \sum_{i \in H} \tilde{D}^i(p).$$

The economy's resource endowment is

$$r = \sum_{i \in H} r^i.$$

The supply side of the unrestricted economy is characterized as

$$S(p) = \sum_{j \in F} S^j(p),$$

and for the artificially restricted economy we have

$$\tilde{S}(p) = \sum_{j \in F} \tilde{S}^j(p).$$

We can now summarize supply, demand, and endowment as an excess demand correspondence.

Definition The excess demand correspondence at prices $p \in P$ is $Z(p) \equiv D(p) - S(p) - \{r\}$.

The excess demand correspondence of the artificially restricted model is $\tilde{Z}(p) = \tilde{D}(p) - \tilde{S}(p) - \{r\}$.

Having defined excess demand, we can now state and prove the Walras' Law, first for the unrestricted economy and then for the artificially restricted economy.

Theorem 17.5 (Walras' Law) Assume C.IV and C.V. Suppose $Z(p)$ is well defined and let $z \in Z(p)$. Then $p \cdot z = 0$.

Proof Let $z \in Z(p)$. Substituting into the definition of $Z(p)$, we have

$$p \cdot z = p \cdot \sum_{i \in H} x^i - p \cdot \sum_{j \in F} y^j - p \cdot \sum_{i \in H} r^i$$
$$\text{for some } x^i \in D^i(p), y^j \in S^j(p).$$

For each $i \in H$, by Lemma 17.4,

$$p \cdot x^i = M^i(p) = p \cdot r^i + \sum_{j \in F} \alpha^{ij} \pi^j(p)$$
$$= p \cdot r^i + \sum_{j \in F} \alpha^{ij} p \cdot y^j.$$

Now summing over $i \in H$, we get

$$\sum_{i \in H} p \cdot x^i = \sum_{i \in H} p \cdot r^i + \sum_{i \in H} \sum_{j \in F} \alpha^{ij} (p \cdot y^j).$$

Taking the vector p outside the sums and reversing the order of summation in the last term yields

$$p \cdot \sum_{i \in H} x^i = p \cdot \sum_{i \in H} r^i + p \cdot \sum_{j \in F} \sum_{i \in H} \alpha^{ij} y^j.$$

Recall that $\sum_{i \in H} \alpha^{ij} = 1$ for each j, and that $r = \sum_{i \in H} r^i$. We have then

$$p \cdot \sum_{i \in H} x^i = p \cdot r + p \cdot \sum_{j \in F} y^j.$$

That is, the value at market prices p of aggregate demand equals the value of endowment plus aggregate supply. Transposing the right-hand side to the left and recalling that $z = \sum_{i\in H} x^i - \sum_{j\in F} y^j - r$, we obtain

$$p \cdot \left[\sum_{i\in H} x^i - \sum_{j\in F} y^j - r \right] = p \cdot z = 0.$$

QED

The Walras' Law tells us that at prices where supply, demand, profits, and income are well defined, planned aggregate expenditure equals planned income from profits and sales of endowment. Hence, the value of planned purchases equals the value of planned sales and the net value at market prices of excess demand is nil. Unfortunately, $Z(p)$ is not always well defined. This arises because Y^j and $B^i(p)$ may be unbounded and hence may not include well-defined maxima of $\pi^j(\cdot)$ or $u^i(\cdot)$, respectively. This shifts our focus to $\tilde{Z}(p)$, which we know to be well defined for all $p \in P$. We now establish the counterpart of the Walras' Law for $\tilde{Z}(p)$.

Theorem 17.6 (Weak Walras' Law) Assume C.IV and C.V. Let $z \in \tilde{Z}(p)$. Then $p \cdot z \leq 0$. Further, if $p \cdot z < 0$ then there is $k = 1, 2, 3, \ldots, N$ so that $z_k > 0$.

Proof $p\cdot z = p\cdot\sum_{i\in H} x^i - p\cdot\sum_{j\in F} y^j - p\cdot\sum_{i\in H} r^i$, where $x^i \in \tilde{D}^i(p)$, $y^j \in \tilde{S}^j(p)$. For each $i \in H$,

$$\begin{aligned} p \cdot x^i \leq \tilde{M}^i(p) &= p \cdot r^i + \sum_{j\in F} \alpha^{ij} \tilde{\pi}^j(p) \\ &= p \cdot r^i + \sum_{j\in F} \alpha^{ij} (p \cdot y^j), \end{aligned}$$

and

$$\begin{aligned} \sum_{i\in H} p \cdot x^i &\leq \sum_{i\in H} p \cdot r^i + \sum_{i\in H} \sum_{j\in F} \alpha^{ij} (p \cdot y^j) \\ p \cdot \sum_{i\in H} x^i &\leq p \cdot \sum_{i\in H} r^i + p \cdot \sum_{j\in F} \sum_{i\in H} \alpha^{ij} y^j. \end{aligned}$$

Note the changed order of summation in the last term. Recall that $\sum_{i\in H} \alpha^{ij} = 1$ for each j and that $r = \sum_{i\in H} r^i$. We have then

$$p \cdot \sum_{i\in H} x^i \leq p \cdot r + p \cdot \sum_{j\in F} y^j.$$

Transposing the right-hand side to the left and recalling that $z =$

$\sum_{i \in H} x^i - \sum_{j \in F} y^j - r$, we get

$$p \cdot \left[\sum_{i \in H} x^i - \sum_{j \in F} y^j - r \right] = p \cdot z \leq 0.$$

The left-hand side in this expression is

$$\sum_{i \in H} [p \cdot x^i] - \sum_{i \in H} [\tilde{M}^i(p)].$$

If $p \cdot z < 0$ then for some $i \in H, p \cdot x^i < \tilde{M}^i(p)$. In that case, by Lemma 17.5, $|x^i| = c$ and hence x^i is not attainable. Unattainability implies $z_k > 0$ for some $k = 1, 2, \ldots, N$. QED

Lemma 17.8 Assume C.I–C.VI, C.VIII, and P.I–P.III. The range of $\tilde{Z}(p)$ is bounded. $\tilde{Z}(p)$ is upper hemicontinuous and convex valued.

Proof $\tilde{Z}(p) = \sum_{i \in H} \tilde{D}^i(p) - \sum_{j \in F} \tilde{S}^j(p) - \{\sum_{i \in H} r^i\}$ is the finite sum of bounded sets and is therefore bounded. It is a finite sum of upper hemicontinuous convex correspondences and is hence convex and upper hemicontinuous. QED

As an artificial construct to allow us to prove the existence of equilibrium in the market economy, we introduce an artificially restricted economy.

17.5 The artificially restricted economy

We will describe the artificially restricted economy by taking the production technology of each firm j to be $\tilde{Y}^j$ rather than Y^j, thus making the supply correspondence $\tilde{S}^j(p)$ rather than $S^j(p)$, and by taking the demand correspondence of each household i to be $\tilde{D}^i(p)$ rather than $D^i(p)$. In this special restricted case we will refer to the excess demand correspondence of the economy as $\tilde{Z}(p)$. By Theorems 17.1 and 17.3, the artificially restricted excess demand correspondence is well defined for all $p \in P$:

$$\tilde{Z}: P \to \mathbf{R}^N.$$

We use the artificially restricted economy above as a mathematical construct, which is convenient because supply, demand, and excess demand are everywhere well defined. The unrestricted economy is defined by Y^j, D^i, and Z. As demonstrated in Theorem 17.1 and Lemma 17.6, $Z(p)$ and $\tilde{Z}(p)$ will coincide for elements of $Z(p)$ corresponding to attainable points in $\tilde{S}^j(p)$

and $\tilde{D}^i(p)$. The set $\tilde{Z}(p)$ is nonempty for all $p \in P$, whereas $Z(p)$ may not be well defined (nonempty) for some elements of $p \in P$.

Recall the following properties of $\tilde{Z}(p)$:

(1) Weak Walras' Law (Theorem 17.6): Assuming P.I - P.IV, C.IV and C.VI, we have $z \in \tilde{Z}(p)$ implies $p \cdot z \leq 0$. Further, if $p \cdot z < 0$ then there is $k = 1, 2, 3, \ldots, N$, so that $z_k > 0$.

(2) $\tilde{Z}(p)$ is well defined for all $p \in P$ and is everywhere upper hemicontinuous and convex valued, assuming C.I–C.VI and P.I–P.IV. This is Theorems 17.1 and 17.3 and lemma 17.8.

We will use these properties to prove the existence of market clearing prices in the artificially restricted economy. We will then use Theorems 17.1 and 17.6 and C.VI to show that the equilibrium of the artificially restricted economy is also an equilibrium of the unrestricted economy. To start the process of establishing the existence of an equilibrium for the artificially restricted economy, we need a price adjustment function. We plan to use the Kakutani Fixed-Point Theorem, and thus we hope to construct an upper hemicontinuous, convex-valued price adjustment correspondence.

Let $\rho(z) \equiv \{p^* | p^* \in P, p^* \cdot z \text{ maximizes } p \cdot z \text{ for all } p \in P\}$. $\rho(z)$ is the price adjustment correspondence. For each excess demand vector z, ρ chooses a price vector based on increasing the prices of goods in excess demand while reducing the prices of goods in excess supply. Choose positive real C so that $|\tilde{Z}(p)| < C$ for all $p \in P$. We know that C exists (by lemma 17.8) since $\#F$ and $\#H$ are finite and each of the $\tilde{D}^i(p)$, $\tilde{S}^j(p)$ is chosen from a bounded set (the set of attainable allocations is bounded by Theorem 8.2). Then let $\Delta = \{x | x \in \mathbf{R}^N, |x| \leq C\}$. Note that Δ is compact and convex:

$$\rho : \Delta \to P$$
$$\tilde{Z} : P \to \Delta.$$

Lemma 17.9 $\rho(z)$ is upper hemicontinuous for all $z \in \Delta$; $\rho(z)$ is convex and nonnull for all $z \in \Delta$.

Proof Exercise 17.6.

17.6 Existence of competitive equilibrium

We are now ready to establish existence of competitive general equilibrium. We focus first on the artificially restricted economy and then extend our results to the unrestricted economy.

Definition $p^0 \in P$ is said to be a competitive equilibrium price vector (of the unrestricted market economy) if there is $z^0 \in Z(p^0)$ so that $z^0 \le 0$ (coordinatewise) and $p^0_k = 0$ for k so that $z^0_k < 0$.

Theorem 17.7 Let the economy fulfill C.I–C.VI, C.VIII, and P.I–P.IV. Then there is a competitive equilibrium p^0 for the economy.

The strategy of proof is to create a grand upper hemicontinuous convex-valued mapping, $\Phi(\cdot)$, from $\Delta \times P$, the Cartesian product of (artificially restricted) excess demand space, Δ, with price space, P, into itself. The mapping takes prices and maps them into the corresponding excess demands and takes excess demands and maps them into corresponding prices. The mapping Φ will have a fixed point by (the corollary to) the Kakutani Fixed-Point Theorem. The fixed point of the price adjustment correspondence, $\rho(\cdot)$, will take place at a market equilibrium of the artificially restricted economy. We will then use Theorems 17.1 and 17.6 and Lemma 17.6 to show that the equilibrium of the artificially restricted economy is also an equilibrium of the original (unrestricted) economy. This follows because the equilibrium of the artificially restricted economy is attainable. Hence, at the artificially restricted economy's equilibrium prices, artificially restricted and unrestricted demands and supplies coincide.

Proof Let $(p, z) \in P \times \Delta, \Phi(p, z) \equiv \{(\bar{p}, \bar{z}) | \bar{p} \in \rho(z), \bar{z} \in \tilde{Z}(p)\}$. Then $\Phi : P \times \Delta \to P \times \Delta$. Φ is nonnull, upper hemicontinuous, and convex valued. $P \times \Delta$ is compact and convex. Then by Corollary 16.1 to the Kakutani Fixed-Point Theorem there is $(p^0, z^0) \in P \times \Delta$ so that (p^0, z^0) is a fixed point of Φ:

$$(p^0, z^0) \in \Phi(p^0, z^0),$$
$$p^0 \in \rho(z^0),$$
$$z^0 \in \tilde{Z}(p^0).$$

We will now demonstrate that (p^0, z^0) represents an equilibrium of the artificially restricted economy. For each $i \in H$, and for each $j \in F$, there is $x^{0i} \in \tilde{D}^i(p^0)$, $y^{0j} \in \tilde{S}^j(p^0)$, so that $x^0 = \sum_i x^{0i}$, $y^0 = \sum_j y^{0j}$, with $z^0 = x^0 - y^0 - r$, and by the Weak Walras' Law, $p^0 \cdot z^0 \le 0$. But p^0 maximizes $p \cdot z^0$ for $p \in P$. This implies $z^0 \le 0$, since if there were any positive coordinate in z^0 then the maximum value of $p \cdot z^0$ would be positive. Moreover, we have either (Case 1) $p^0 \cdot z^0 = 0$ (in which case it follows that $z^0 = 0$ or $z^0_k < 0$ implies $p^0_k = 0$) or (Case 2) $p^0 \cdot z^0 < 0$ (in which case the Weak Walras' Law implies $z_k^0 > 0$ some k). But in Case 2, max $p \cdot z^0$ would then be positive, which is a contradiction. Hence Case 2 cannot arise and we have $p^0 \cdot z^0 = 0$, with either $z^0 = 0$ or if for some $k, z_k^0 < 0$, then $p_k^0 = 0$. This establishes (p^0, z^0) as an equilibrium for the artificially restricted economy. Now we must demonstrate that it is an equilibrium for the unrestricted economy as well. We have

$$z^0 = x^0 - y^0 - r$$

or

$$x^0 - z^0 = y^0 + r.$$

Since $z^0 \leq 0, x^0 - z^0 \geq x^0 \geq 0$. Thus $y^0 + r \geq 0$. Therefore, y^0 is attainable; this implies, by Theorem 17.1, that $y^{0j} \in S^j(p^0)$ for all $j \in F$. Furthermore, since $y^0 + r \geq x^0, x^0$ is attainable. Hence, by Lemma 17.7, $x^{0i} \in D^i(p^0)$ for all $i \in H$. Thus we have $p^0 \in P, y^{0j} \in S^j(p^0)$, and $x^{0i} \in D^i(p^0)$, so that $\sum_{i \in H} x^{0i} - \sum_{j \in F} y^{0j} - \sum_{i \in H} r^i \leq 0$, with $p_k = 0$ for all k such that $z_k^0 < 0$. Hence (p^0, z^0) is an equilibrium for the unrestricted economy. QED

Theorem 17.7 completes the treatment of the existence of equilibrium with set-valued demand and supply behavior. We have demonstrated that all of the results on continuity of demand and supply and existence of equilibrium demonstrated for continuous point-valued demand and supply have counterparts in upper hemicontinuous convex-valued demand and supply. The essential elements that carry over are continuity and convexity in both settings. Note that because the efficiency results of Chapter 12 nowhere depend on point valuedness of demand or supply they are immediately applicable to the correspondence-valued demand and supply behavior studied here in Chapter 17.

17.7 Bibliographic note

The use of set-valued supplies and demands in a general equilibrium model, allowing for flat segments in preferences and technologies, first appears in Arrow and Debreu (1954). It is thoroughly expounded in Debreu (1959).

Exercises

17.1 Prove Lemma 17.3 (homogeneity of degree 0): Assume P.I–P.IV. Let $\lambda > 0$, $p \in \mathbf{R}_+^N$. Then $\tilde{S}^j(\lambda p) = \tilde{S}^j(p)$ and $S^j(\lambda p) = S^j(p)$.

17.2 Prove part of Theorem 17.2: Assume P.II, P.III, C.I, C.III, and C.VIII. Then $\tilde{B}^i(p) \cap X^i$ is upper hemicontinuous for all $p \in P$.

17.3 Prove Theorem 17.4: Assume P.II, P.III, C.I, C.III, C.V, C.VI, and C.VIII. Then $\tilde{B}^i(p)$ and $\tilde{D}^i(p)$ are convex valued.

17.4 Prove Lemma 17.4: Under C.IV and C.V, $x \in D^i(p)$ implies $p \cdot x = M^i(p)$.

17.5 Prove Lemma 17.5: Under C.IV and C.V, if $x \in \tilde{D}^i(p)$ and $p \cdot x < \tilde{M}^i(p)$, then $|x| = c$. This result follows from weak monotonicity (C.IV). See the proof of Theorem 6.2.

17.6 Prove Lemma 17.9: $\rho(\cdot)$ is upper hemicontinuous throughout Δ; $\rho(z)$ is convex and nonnull for any $z \in \Delta$.

17.7 The Arrow Corner is a failure of lower hemicontinuity of the budget correspondence and of upper hemicontinuity of the demand correspondence. It occurs when some prices are zero and when income is just sufficient to achieve the boundary of the consumption set X^i (in a typical example, this will occur at a zero income where X^i is the nonnegative orthant). Consider the following example. Let $N = 2, X^i = \mathbf{R}^2_+$, and

$$p^\nu = (1 - 1/\nu, 1/\nu), \nu = 1, 2, 3, \ldots.$$

Then we have $p^\nu \to p^0 = (1, 0)$. Let c (the bound on the size of the demand vector) be chosen so that $100 < c < \infty$. Let household i's endowment vector r^i equal $(0, 100)$, with sale of r^i being i's sole source of income. Then we have

$$\tilde{B}^i(p) = \{x | x = (x_1, x_2), |x| \leq c, p \cdot x \leq p \cdot r^i\}.$$

Let i's utility function be $u^i(x_1, x_2) = x_1 + x_2$ so that $\tilde{D}^i(p) = \{x' | x' \in \tilde{B}^i(p) \cap \mathbf{R}^2_+, x'$ maximizes $u^i(x)$ for all $x \in \tilde{B}^i(p) \cap \mathbf{R}^2_+\}$. Demonstrate the following points:

(i) Show that $(0, c) \in \tilde{B}^i(p^0)$.

(ii) Show that $x \in \tilde{B}^i(p^\nu), x = (x_1, x_2)$, implies $x_2 \leq 100$.

(iii) Show that $\tilde{D}^i(p^0) = \{(0, c)\}$

(iv) Show that $\tilde{B}^i(p)$ is not lower hemicontinuous at $p = p^0$.

(v) Show that $\tilde{D}^i(p)$ is not upper hemicontinuous at $p = p^0$.

Discuss this example with regard to the Maximum Theorem (Theorem 16.2).