11

General equilibrium of the market economy: The unbounded technology case

11.1 General equilibrium

In this chapter we will consider the general equilibrium of an economy with (possibly) unbounded production technologies where demands and supplies are point valued. We will establish the most important single result in this book, Theorem 11.1, the existence of general equilibrium. We know that a sufficient condition for point-valuedness of supply and demand is strict convexity of tastes and technologies, P.V and C.VII. As noted in Chapter 6, homogeneity of degree zero of $D^i(\cdot)$ and $S^j(\cdot)$ in p means that we may, without loss of generality, restrict the price space to be the unit simplex in $\mathbf{R}^N$,

$$P = \left\{ p \mid p \in \mathbf{R}^N, p_k \geq 0, k = 1 \ldots, N, \sum_{k=1}^{N} p_k = 1 \right\}.$$

From Chapter 10, the market excess demand function is defined as

$$Z(p) = \sum_{i \in H} D^i(p) - \sum_{j \in F} S^j(p) - r.$$

There are some regions of P where $Z(\cdot)$ may not be well defined, since the maximization of profits in determining S^j or utility in determining D^i may not have a well-defined value. This arises in as much as the opportunity sets, Y^j or $B^i(p)$, may be unbounded. Then profit or utility may lack a well-defined maximum.

We are interested in investigating a market clearing equilibrium defined as:

Definition $p^0 \in P$ is said to be an equilibrium price vector if $Z(p^0) \leq 0$ (the inequality applies coordinatewise) with $p^0_k = 0$ for k such that $Z_k(p^0) < 0$.

That is, an equilibrium is characterized by market clearing for all goods except perhaps free goods, which may be in excess supply in equilibrium.

11.2 An artificially restricted economy

We would like to establish the existence of a general equilibrium where the economy is characterized by the excess demand function $Z(p)$. We are hampered in this quest because there are regions of price space where $Z(p)$ is not well defined. The strategy of proof is to consider the bounded counterpart of the economy, the artificially bounded economy characterized by the excess demand function $\tilde{Z}(p)$. We will establish the existence of equilibrium in this artificially bounded economy. We know we can do so, since this economy fulfills all of the conditions required of an economy in Chapters 4–7, particularly Theorem 7.1. There will be an equilibrium price vector p^* for the artificially bounded economy so that $\tilde{Z}(p^*) \leq 0$. But the equilibrium allocation is attainable. As noted in Lemma 7.1, the quantity constraint on $\tilde{D}^i(p)$ is not binding in equilibrium. By Theorems 8.3(b) and 9.1(b), $S^j(p^*) = \tilde{S}^j(p^*)$ and $D^i(p^*) = \tilde{D}^i(p^*)$. At the equilibrium of the artificially bounded economy, demand and supply coincide with those of the unrestricted economy. Therefore, $\tilde{Z}(p^*) = Z(p^*) \leq 0$. But then the trick is done. We have an equilibrium for the original economy characterized by $Z(p)$.

We will describe the artificially bounded economy by taking the production technology of each firm j to be $\tilde{Y}^j$ rather than Y^j and by taking the demand function of each household i to be $\tilde{D}^i(p)$ rather than $D^i(p)$. In this special restricted case we will refer to the excess demand function of the economy as $\tilde{Z}(p)$. As demonstrated in Chapters 4, 5, and 6, the artificially restricted excess demand function is well defined for all $p \in P$. $\tilde{Z}:P \to \mathbf{R}^N$. The unrestricted economy is defined by Y^j, D^i, and Z. As demonstrated in Chapters 8–10, $Z(p)$ and $\tilde{Z}(p)$ will coincide for p so that each firm and household's plans in the restricted economy, $\tilde{S}^j(p)$ and $\tilde{D}^i(p)$, are attainable.

We have the following observations on $\tilde{Z}(p)$:

Weak Walras' Law (Theorem 6.2): For all $p \in P, p \cdot \tilde{Z}(p) \leq 0$. For p such that $p \cdot \tilde{Z}(p) < 0$, there is $k = 1, 2, \ldots, N$, so that $\tilde{Z}_k(p) > 0$.

$\tilde{Z}(p)$ is a continuous function, assuming P.II–P.V, C.I–C.V, C.VII, and C.VIII (Theorem 4.1, Theorem 5.2).

From Chapter 7 we know that there is $p^0 \in P$, so that p^0 is an equilibrium of the artificially restricted economy characterized by $\tilde{Z}(p)$. How do we know this? The economy characterized by $\tilde{Y}^j$ and $\tilde{D}^i(p)$ fulfills all of the assumptions of Theorem 7.1 when we substitute $\tilde{Y}^j$, the bounded subset of Y^j, for $\mathcal{Y}^j$, the bounded technology sets of Chapters 4 through 7. Therefore, by applying Theorem 7.1 we can find $p^0 \in P$ so that $\tilde{Z}(p^0) \leq 0$, with $p^0_k = 0$ for k so that $\tilde{Z}_k(p^0) < 0$.

11.3 General equilibrium of the unrestricted economy

We now wish to establish the existence of general equilibrium in the unrestricted economy, Theorem 11.1. We start with Lemma 11.1: Consider the restricted economy characterized by $\tilde{Y}^j$, $\tilde{S}^j$, and $\tilde{D}^i$ and show that it has a general equilibrium by Theorem 7.1. This result is in itself of no interest since the economy to which it applies is entirely artificial. We will then show that the equilibrium of the artificially restricted economy is attainable in the actual economy. It then follows that, at the equilibrium prices of the artificially restricted economy, the firm supply functions and household demand functions of the actual economy coincide with those of the restricted economy. This coincidence follows from Theorem 8.3(b) and Theorem 9.1(b). Hence, the equilibrium price vector developed in Lemma 11.1 is also an equilibrium of the unrestricted economy. This proves Theorem 11.1.

Lemma 11.1 [1] Assume P.II–P.V, C.I–C.V, CVII, and C.VIII. There is $p^* \in P$ so that p^* is an equilibrium of the artificially restricted economy. That is $\tilde{Z}(p^*) \leq 0$ and $p^*_k = 0$ for k so that $\tilde{Z}_k(p^*) < 0$.

Proof Lemma 11.1 is merely a restatement of Theorem 7.1, so the proof is completely redundant. We reproduce the treatment here merely for completeness.

We will show that there is $p^* \in P$ so that $\tilde{Z}(p^*) \leq 0$, with $p^*_k = 0$, for k so that $\tilde{Z}_k(p^*) \leq 0$. p^* is an equilibrium price vector of the artificially bounded economy. We formulate a price adjustment function, $T : P \to P$. Define $T(p)$ in the following fashion for each coordinate $k = 1, 2, 3, \ldots, N$:

$$T_k(p) \equiv \frac{p_k + \max[0, \tilde{Z}_k(p)]}{1 + \sum_{n=1}^{N} \max[0, \tilde{Z}_n(p)]} = \frac{p_k + \max[0, \tilde{Z}_k(p)]}{\sum_{n=1}^{N} \{p_n + \max[0, \tilde{Z}_n(p)]\}}.$$

The price adjustment function T raises the relative price of goods in excess demand and reduces that of goods in excess supply while keeping the price

[1] Thanks to John Roemer and Li Li for help in formulating the proof.

vector on the simplex. The price adjustment function here is specified differently — in this more general setting — from the one used in Chapter 1, to assure that the denominator (trivially) be positive.

By Lemma 6.1, $\tilde{Z}(p)$ is a continuous function. Then $T(p)$ is a continuous function from the simplex into itself since continuity is preserved under the operations of max, addition, and division by a positive-valued continuous function. An illustration of the notion of a continuous function from P into P is presented in Figure 7.1. By the Brouwer Fixed-Point Theorem there is $p^* \in P$ so that $T(p^*) = p^*$. But then for all $k = 1, \ldots, N$,

$$T_k(p^*) = p_k^* = \frac{p_k^* + \max[0, \tilde{Z}_k(p^*)]}{1 + \sum_{n=1}^{N} \max[0, \tilde{Z}_n(p^*)]}.$$

Thus, either Case 1 or Case 2 below applies.

CASE 1 $p_k^* = 0 = \max[0, \tilde{Z}_k(p^*)]$. Hence $\tilde{Z}_k(p^*) \leq 0$.

CASE 2

$$p_k^* = \frac{p_k^* + \max[0, \tilde{Z}_k(p^*)]}{1 + \sum_{n=1}^{N} \max[0, \tilde{Z}_n(p^*)]} > 0.$$

To avoid repeated tedious notation, let

$$0 < \alpha = \frac{1}{1 + \sum_{n=1}^{N} \max[0, \tilde{Z}_n(p^*)]} \leq 1.$$

We have

$$T_k(p^*) = p_k^* = \alpha(p_k^* + \max[0, \tilde{Z}_k(p^*)])$$

$$p_k^* = \alpha p_k^* + \alpha \max[0, \tilde{Z}_k(p^*)]$$

or

$$(1 - \alpha)p_k^* = \alpha \max[0, \tilde{Z}_k(p^*)].$$

Multiplying through by $\tilde{Z}_k(p^*)$, we get

$$(1 - \alpha)p_k^* \tilde{Z}_k(p^*) = \alpha(\max[0, \tilde{Z}_k(p^*)])\tilde{Z}_k(p^*) \qquad (*).$$

We can restate the Weak Walras' Law as

$$\begin{aligned} 0 \geq p^* \cdot \tilde{Z}(p^*) &= \sum_{k \in \text{Case1}} p_k^* \tilde{Z}_k(p^*) + \sum_{k \in \text{Case2}} p_k^* \tilde{Z}_k(p^*) \\ &= 0 + \sum_{k \in \text{Case2}} p_k^* \tilde{Z}_k(p^*) \\ &= \sum_{k \in \text{Case2}} p_k^* \tilde{Z}_k(p^*). \end{aligned}$$

Multiplying through by $(1-\alpha)$, and substituting per (*) we get

$$0 \geq (1-\alpha) \sum_{k \in \text{Case2}} p_k^* \tilde{Z}_k(p^*) = \alpha \sum_{k \in \text{Case2}} (\max[0, \tilde{Z}_k(p^*)]) \tilde{Z}_k(p^*).$$

Then (strictly) positive entries in the sum on the right-hand-side will occur for $k \in$ Case 2 so that $\tilde{Z}_k(p^*) > 0$. Thus

$$0 \geq \sum_{k \in \text{Case2}, \tilde{Z}_k(p^*) > 0} [\tilde{Z}_k(p^*)]^2.$$

But this means that $\tilde{Z}_k(p^*) \leq 0$, for all k in Case 2. However, then, there is no k, either in Case 1 or 2, so that $\tilde{Z}_k(p^*) > 0$. From the Weak Walras' Law it follows that $p^* \cdot \tilde{Z}(p^*) = 0$. Hence for k so that $\tilde{Z}_k(p^*) < 0$, it follows that $p_k^* = 0$. This completes the proof. QED

Theorem 11.1 Assume P.II–P.V, C.I–C.V, CVII, and C.VIII. There is $p^* \in P$ so that p^* is an equilibrium price vector. That is, $Z(p^*) \leq 0$ and $p_k^* = 0$ for k so that $Z_k(p^*) < 0$.

Proof We note from Lemma 11.1 that there is an equilibrium price vector $p^* \in P$ for the artificially restricted economy. There is $p^* \in P$ so that $\tilde{Z}(p^*) \leq 0$ with $p_k^* = 0$ for k so that $\tilde{Z}_k(p^*) < 0$. Now we must show that the equilibrium of the restricted economy is also an equilibrium of the unrestricted economy. First we note that the production plans at p^* of each firm in the artificially restricted economy, $\tilde{S}^j(p^*)$, are attainable, and similarly for $\tilde{D}^i(p^*)$. This follows simply from the definition of equilibrium, which implies that the equilibrium allocation be attainable. That is, $\tilde{Z}(p^*) = \sum_{i \in H} \tilde{D}^i(p^*) - r - \sum_{j \in F} \tilde{S}^j(p^*) \leq 0$ implies $r + \sum_{j \in F} \tilde{S}^j(p^*) \geq \sum_{i \in H} \tilde{D}^i(p^*) \geq 0$. But then by Theorem 8.3(b) we have $\tilde{S}^j(p^*) = S^j(p^*)$ for all $j \in F$. It follows that $\tilde{\pi}^j(p^*) = \pi^j(p^*)$ and hence $\tilde{M}^i(p^*) = M^i(p^*)$ for all $i \in H$. But then by Theorem 9.1(b), $\tilde{D}^i(p^*) = D^i(p^*)$. By definition, $Z(p^*) = \sum_{i \in H} D^i(p^*) - r - \sum_{j \in F} S^j(p^*)$. Therefore, $\tilde{Z}(p^*) = Z(p^*)$. But then $Z(p^*) \leq 0$, with $p_k^* = 0$ for k so that $Z_k(p^*) < 0$, so p^* is an equilibrium price vector. QED

Theorem 11.1 is the most important single result of this book. It says that the competitive economy, guided only by prices, has a market-clearing equilibrium outcome. The decentralized price-guided economy has a consistent solution. This is the defining result of the general equilibrium theory.

11.4 The Uzawa Equivalence Theorem

The principal mathematical tool we used in proving Lemma 11.1 and hence Theorem 11.1 is the Brouwer Fixed-Point Theorem. There is now a distinctive result that shows that the use of the Brouwer Fixed-Point Theorem is not merely convenient. It is essential. We will demonstrate the mathematical equivalence of two propositions: (i) the existence of equilibrium in an economy characterized by a continuous excess demand function fulfilling Walras' Law and (ii) the Brouwer Fixed-Point Theorem. We already know that the Brouwer Fixed-Point Theorem implies existence of equilibrium. We will now demonstrate the converse: If we are always sure of existence of equilibrium in such an economy, then the Brouwer Fixed-Point Theorem must follow. The Brouwer Fixed-Point Theorem implies existence of general equilibrium; existence of general equilibrium implies the Brouwer Fixed-Point Theorem. Thus, the two apparently distinct results are mathematically equivalent.

We will now demonstrate this equivalence. Just to get terminology and notation straight (and to keep it distinct from the economic model developed above) we will restate some results and introduce some new notation for familiar constructs.

Let S ($\equiv P$) be the unit simplex in $\mathbf{R}^N$. Recall two propositions:

Brouwer Fixed-Point Theorem (BFPT) Let $f: S \to S$, where f is continuous. Then there is $p^* \in S$ so that $p^* = f(p^*)$.

Walrasian Existence of Equilibrium Proposition (WEEP) Let $X: S \to \mathbf{R}^N$ so that

(1) $X(p)$ is continuous for all $p \in S$ and
(2) $p \cdot X(p) = 0$ (Walras' Law) for all $p \in S$.[2]

Then there is $p^* \in S$ so that $X(p^*) \leq 0$ with $p_i^* = 0$ for i so that $X_i(p^*) < 0$.

The observation that these two results are equivalent constitutes Theorem 11.2, below. Mathematical equivalence means that each proposition implies

[2] We use the strong form of Walras' Law for convenience.

the other. We already know that BFPT implies WEEP; that was Theorem 1.2. It remains to demonstrate that the implication goes the other way as well. The proposition requires that – using WEEP but not BFPT – we prove that for an arbitrary continuous function from the simplex to itself, there is a fixed point. The strategy of proof is to take an arbitrary continuous function $f(p)$ from the simplex into itself. We use $f(p)$ to construct a continuous function mapping from S into $\mathbf{R}^N$ fulfilling Walras' Law. That is, we construct an "excess demand" function (derived from no actual economy but fulfilling the properties required in WEEP). The strategy of proof then is to find the general equilibrium price vector associated with this excess demand function and show that it is also a fixed point for the original function. Obviously, this plan requires clever construction of the excess demand function.

Theorem 11.2 (Uzawa Equivalence Theorem) [3] WEEP implies BFPT.

Proof We must demonstrate the following property: Let $f(\cdot)$ be an arbitrary continuous function mapping S into S. Assume WEEP but not BFPT. Then there is $p^* \in S$ so that $f(p^*) = p^*$.

Let $f{:}S \to S$, where f is continuous. Let

$$\begin{aligned}\mu(p) &\equiv \frac{p \cdot f(p)}{|p|^2} \\ &\equiv \frac{|p||f(p)|}{|p|^2} \cos(p, f(p)) \leq \frac{|f(p)|}{|p|},\end{aligned}$$

where $\cos(p, f(p))$ denotes the cosine of the angle included by $p, f(p)$. Let

$$X(p) \equiv f(p) - \mu(p)p.$$

The function $X(p)$ represents "excess demand." If we have constructed it cleverly enough, the equilibrium price vector of $X(p)$ will also be a fixed point of $f(\cdot)$. The geometry of this construction is illustrated in Figure 11.1. It makes for a compelling visual demonstration that the equilibrium price vector of the excess demand function $X(p)$ is necessarily a fixed point

[3] The result is due to Hirofumi Uzawa (1962).

11.1.eps

Fig. 11.1. The Uzawa Equivalence Theorem

of the function $f(p)$. Note that $X(p)$ fulfills (1) and (2) of WEEP. We have

$$p \cdot X(p) = p \cdot f(p) - \frac{p \cdot f(p)}{|p|^2}|p|^2 = 0;$$

this is Walras' Law (2).

Hence, assuming WEEP, there is $p^* \in S$ so that $X(p^*) \leq 0$. Note that by construction $X(p^*) = 0$. This follows since $p_i^* = 0$ for $X_i(p^*) < 0$. If there were i so that $X_i(p^*) < 0$, it would lead to a contradiction: $p_i^* = 0$, so $0 > X_i(p^*) = f_i(p^*) - \mu(p^*)p_i^* = f_i(p^*) - 0 \geq 0$. Therefore, $X(p^*) = f(p^*) - \mu(p^*)p^* = 0$. Thus $f(p^*) = \mu(p^*)p^*$. But p^* and $f(p^*)$ are both points of the simplex. The only scalar multiple of a point on the simplex that remains on the simplex occurs when the scalar is unity. That is, $f(p^*) \in S$, $p^* \in S$, and $f(p^*) = \mu(p^*)p^*$ implies $\mu(p^*) = 1$, which implies $f(p^*) = p^*$.[4]
QED

What are we to make of the Uzawa Equivalence Theorem? It says that use of the Brouwer Fixed-Point Theorem is not merely one way to prove the existence of equilibrium. In a fundamental sense, it is the only way. Any alternative proof of existence will include, inter alia, an implicit proof of the Brouwer Theorem. Hence this mathematical method is essential; one cannot pursue this branch of economics without the Brouwer Theorem. If Walras failed to provide an adequate proof of existence of equilibrium himself, it was in part because the necessary mathematics was not yet available.

11.5 Bibliographic note

The proof of existence of equilibrium presented here parallels that of Arrow and Debreu (1954). The Uzawa Equivalence Theorem appeared first in Uzawa (1962) and is discussed in Debreu (1982).

Exercises

11.1 Describe the significance of:
   (a) The Uzawa Equivalence Theorem, Theorem 11.2. Does it have an implication for the importance of mathematics in economics?
   (b) The Existence of General Equilibrium Theorem, Theorem 1.2, Theorem 7.1, or Theorem 11.1.

[4] My acknowledgment and thanks go to Jin-lung Lin for providing the central idea of this argument.

(c) Consider the following definition:

$$\{p^0, x^{0i}, y^{0j}\}, p^0 \in \mathbf{R}_+^N, i \in H, j \in F,$$

is said to be a competitive equilibrium if

(i) $y^{0j} \in Y^j$ and $p^0 \cdot y^{0j} \geq p^0 \cdot y$ for all $y \in Y^j$, for all $j \in F$,

(ii) $x^{0i} \in X^i$, $p^0 \cdot x^{0i} \leq M^i(p^0) = p^0 \cdot r^i + \sum_{j \in F} \alpha^{ij} p^0 \cdot y^{0j}$ and $x^{0i} \succeq_i x$ for all $x \in X^i$ with $p^0 \cdot x \leq M^i(p^0)$ for all $i \in H$, and

(iii) $0 \geq \sum_{i \in H} x^{0i} - \sum_{j \in F} y^{0i} - \sum_{i \in H} r^i$ with $p_k^0 = 0$ for coordinates k so that the strict inequality holds.

(a) The concept of competitive equilibrium is supposed to reflect decentralization of economic behavior. Explain how this definition embodies the concept of decentralization.

(b) The concept of competitive equilibrium is supposed to reflect market clearing. Explain how this definition includes market clearing.

11.2 Consider the general competitive equilibrium of a production economy with redistributive taxation of income from endowment. Half of each household's income from endowment (based on actual endowment, not net sales) is taxed away. The proceeds of the tax are then distributed equally to all households. We then have,

$$M^i(p) = p \cdot (.5r^i) + \sum_{j \in F} \alpha^{ij} p \cdot y^j + T,$$

where T is the transfer of tax revenues to the household,

$$T = (1/\#H) \sum_{h \in H} p \cdot (.5r^h).$$

(a) Define a competitive equilibrium in this economy.

(b) State Walras' Law for this economy. Does it hold? Explain.

(c) Does a competitive equilibrium generally exist in this economy? Explain.

11.3 We now consider the economy of Exercise 11.3 with income taxation only on the household's net sales of endowment (rather than on the value of the full endowment as above). Household i's after tax income is now

$$M^i(p) = (1 - \tau) p \cdot (r^i - x^i)_+ + T,$$

where the notation $(\cdot)_+$ indicates the vector consisting of the nonnegative coordinates of $(\cdot)$ with zeros replacing the negative coordinates of $(\cdot)$. We now let $T = (1/H)\tau \sum_{i=1}^{H} (p \cdot (r^i - x^i)_+)$.

(a) Define a competitive equilibrium in this economy.
(b) State Walras' Law for this economy. Does it hold? Explain.
(c) Does a competitive equilibrium generally exist in this economy? Explain.

11.4 The model below is an interpretation of E. Malinvaud's Theory of Unemployment Reconsidered.

Consider the general equilibrium of a private ownership production economy. There are $\#H$ households, $i = 1, \ldots, \#H$. Each household i has a continuous monotonic, concave utility function $u^i(\cdot)$ and is endowed with resources $r^i \in \mathbf{R}^N_+$. There is a finite number of firms comprising the set F. Firm j has a compact convex technology set Y^j. Firm supply behavior is guided by simple profit maximization:

$$y^j = \arg\max_{y \in Y^j} p \cdot y,$$

The expression "$y^j = \arg\max_{y \in Y^j} p \cdot y$" defines y^j as the maximizer of $p \cdot y$ in Y^j. i's income is

$$M^i(p) = p \cdot r^i.$$

Note that M^i makes no allowance for the payment of firm profits to owners. i's consumption behavior is

$$(C) \text{ choose } x^{0i} \in \mathbf{R}^N_+, p \cdot x^{0i} \le M^i(p), u^i(x^{0i}) \ge u^i(x)$$
$$\text{for all } x \text{ such that } p \cdot x \le M^i(p).$$

(a) Is Walras' Law fulfilled in the economy in this case? Explain.
(b) Is the excess demand function continuous in prices? Explain briefly. Feel free to cite known results.
(c) Does a competitive general equilibrium exist in the economy? Always? Never? Explain.