# 10

# A market economy: The unbounded technology case

## 10.1 Firms and households

We now bring the elements of Chapters 8 and 9 together to describe the market economy and to develop the Walras' Law. As before, the economy is characterized by the agents in it: households (the set $H$) and firms (the set $F$). A household $i \in H$ is characterized by its endowment of goods $r^i \in \mathbf{R}^N_+$, by its endowed share $\alpha^{ij}$ of firms $j \in F$, and by $\succeq_i$. We assume $\sum_{i \in H} \alpha^{ij} = 1$ for each $j \in F$ and $\alpha^{ij} \geq 0$ for all $i \in H$, $j \in F$. Each firm is 100% owned by one or more shareholders, and there is no negative ownership (no short sales). The initial resource endowment of the economy, designated $r \in \mathbf{R}^N_+$ is $r \equiv \sum_{i \in H} r^i$.

## 10.2 Profits

A firm $j \in F$ is characterized by its possible production technology set $Y^j$. Firm $j$'s profit function is $\pi^j(p) = \max_{y \in Y^j} p \cdot y = p \cdot S^j(p)$. Considering that we need to discuss artificially restricted firm technology sets $\tilde{Y}^j$, it is convenient to have a concept of the profit function for the firm so restricted,

$$\tilde{\pi}^j(p) = \max_{y \in \tilde{Y}^j} p \cdot y = p \cdot \tilde{S}^j(p).$$

Note that the definition of $\tilde{\pi}^j(p)$ is identical to the corresponding definition in Chapter 6 with $\tilde{Y}^j$ substituted for $\mathcal{Y}^j$. Because the formal properties of these sets are the same, the profit functions $\tilde{\pi}^j(p)$ have the same properties.

Theorem 10.1 Assume P.II and P.III. Then $\pi^j(p) \geq 0$ for all $j \in F$, all $p \in \mathbf{R}^N_+$ such that $\pi^j(p)$ is well defined. $\pi^j(p)$ is a continuous function of $p$ in every neighborhood such that $\pi^j(p)$ exists. $\tilde{\pi}^j(p)$ is a well-defined continuous function of $p$ for all $p \in P$, and $\pi^j(p) = \tilde{\pi}^j(p)$ for all $p$ so that

$S^j(p)$ is attainable in $Y^j$; $\pi^j(p) = \tilde{\pi}^j(p)$ for all $p$ so that $S^j(p) = \tilde{S}^j(p)$.

Proof Exercise 10.2.

Note that in Theorem 10.1 $\pi^j(p)$ may not be well defined (may not exist) for some values of $p$. This reflects that since $\pi^j(p)$ is defined as the maximum of a real-valued function on the domain $Y^j$, a well-defined value of $\pi^j(p)$ depends on that maximum existing. Since $Y^j$ is not compact, the maximum may not exist. That is why we depend so heavily on $\tilde{\pi}^j(p)$, defined by the compact domain $\tilde{Y}^j$.

## 10.3 Household income

Household $i$'s income is defined as

$$M^i(p) = p \cdot r^i + \sum_{j \in F} \alpha^{ij} \pi^j(p).$$

That is, we define household income as the sum of the value of the household endowment plus the value of the household's share of firm profits. For the model with restricted firm supply behavior, household income is

$$\tilde{M}^i(p) = p \cdot r^i + \sum_{j \in F} \alpha^{ij} \tilde{\pi}^j(p).$$

Note that $M^i(p)$ is a continuous, nonnegative, real-valued function of $p$ wherever $\pi^j$ is well defined for all $j \in F$. $\tilde{M}^i(p)$ is continuous, real-valued, nonnegative, and well defined for all $p \in P$. $M^i(p) = \tilde{M}^i(p)$ whenever $S^j(p) = \tilde{S}^j(p)$ for all $j \in F$, in particular for $p$ so that $S^j(p)$ is attainable in $Y^j$ for all $j \in F$.

## 10.4 Excess demand and Walras' Law

Definition The excess demand function at prices $p \in P$ is $Z(p) = D(p) - S(p) - r$.

As before we denote $\tilde{Z}(p) = \tilde{D}(p) - \tilde{S}(p) - r$. In the present setting, $\tilde{Z}(p)$ is something of an artificial construct, representing the excess demand function of an economy characterized by artificial bounds on the firms' production technology and households' budget sets of the underlying true economy.

Lemma 10.1 Let $M^i(p)$ and $D^i(p)$ be well defined and assume C.II and C.IV. Then $p \cdot D^i(p) = M^i(p)$.

Proof Exercise 10.3.

Lemma 10.1 develops one of the principal implications of monotone preferences, C.IV. All budgets will be fully spent. This is an essential point in proving Walras' Law. A naive reading of Lemma 10.1 would suggest that it says there is no saving. However, in a model with dated goods, saving takes the form of purchasing goods dated for future delivery.

Walras' Law is one of the essential building blocks of the proof of existence of general equilibrium. It says that at any prices where excess demand is well defined, the value of excess demand, evaluated at prevailing prices, is zero. This is not an equilibrium condition. It is true at all price vectors, in and out of equilibrium. The Walras' Law reflects two essential elements of the model: the disbursement of profits to shareholders (embodied in the definition of the budget constraint) and the equality of expenditure to income (Lemma 10.1—deriving from monotonicity). The first of these is essentially an accounting consistency requirement; the profits have to go somewhere. Monotonicity of preferences reflects the idea of scarcity, which is essential to economic analysis. The Walras' Law then embodies the technical implications of these economic assumptions.

Theorem 10.2 (Walras' Law) Assume C.II and C.IV, and let $Z(p)$ be well defined and point valued[1]. Then $p \cdot Z(p) = 0$.

Proof Note that

$$p \cdot Z(p) = p \cdot \sum_{i \in H} D^i(p) - p \cdot \sum_{j \in F} S^j(p) - p \cdot \sum_{i \in H} r^i.$$

By Lemma 10.1, we have

$$\begin{aligned} p \cdot D^i(p) = M^i(p) &= p \cdot r^i + \sum_{j \in F} \alpha^{ij} \pi_j(p) \\ &= p \cdot r^i + \sum_{j \in F} \alpha^{ij} (p \cdot S^j(p)). \end{aligned}$$

It follows then that

$$\sum_{i \in H} p \cdot D^i(p) = \sum_{i \in H} p \cdot r^i + \sum_{i \in H} \sum_{j \in F} \alpha^{ij} (p \cdot S^j(p)),$$

[1] "Well defined" depends on $p \in P$ being a value where firm profits and supplies are well-defined (exist). Point-valued follows from strict convexity P.V and C.V.

which can be written as

$$p \cdot \sum_{i \in H} D^i(p) = p \cdot \sum_{i \in H} r^i + p \cdot \sum_{j \in F} \sum_{i \in H} \alpha^{ij} S^j(p).$$

Note the changed order of summation in the last term. Recall that $\sum_{i \in H} \alpha^{ij} = 1$ for each $j$. We have then

$$p \cdot \sum_{i \in H} D^i(p) = p \cdot r + p \cdot \sum_{j \in F} S^j(p)$$

$$p \cdot \left[ \sum_{i \in H} D^i(p) - \sum_{j \in F} S^j(p) - r \right] = p \cdot Z(p) = 0.$$

QED

We showed in Chapter 8 that under P.I–P.III and P.IV the attainable subset of $Y^j$ is bounded. We defined $\tilde{Y}^j$ as a bounded subset of $Y^j$ containing the attainable part of $Y^j$ as a proper subset. Under P.I–P.IV it is then redundant to assume P.VI (boundedness) explicitly since it is implied by P.I–P.IV according to Theorems 8.1 and 8.2. The following results from Chapter 6 were proved using P.I–P.III, and P.V. They are still valid and applicable under the definitions of Chapters 8–10, with the understanding that P.IV implies P.VI.

Lemma 6.1 Assume C.I–C.V, C.VII, C.VIII, P.II, P.III, P.V, and P.VI. The range of $\tilde{Z}(p)$ is bounded. $\tilde{Z}(p)$ is continuous and well defined for all $p \in P$.

Theorem 6.2 (Weak Walras' Law) Assume C.I–C.V, C.VII, C.VIII, P.II, P.III, P.V, and P.VI. For all $p \in P$, $p \cdot \tilde{Z}(p) \leq 0$. For $p$ such that $p \cdot \tilde{Z}(p) < 0$, there is $k = 1, 2, \ldots, N$ so that $\tilde{Z}_k(p) > 0$.

The Weak Walras' Law tells us that any value of the truncated demand function $\tilde{Z}(p)$ will have one of two characteristics. Either the value of excess demand, evaluated at prevailing prices, is nil (as in the Walras' Law) or the value is negative and there is positive excess demand for some one or several of the $N$ goods. This differs from the usual Walras' Law (Theorem 10.2) since the excess demand function $\tilde{Z}(p)$ here is based on household demand functions $\tilde{D}^i(p)$ and the firm supply functions $\tilde{S}^j(p)$ that include a restriction to keep demand and supply inside a sphere of radius $c$. The Weak Walras' Law presents the counterpart to Walras' Law we can expect in the truncated version of the model where households may not fully spend income and firms may not fully pursue profitable production if the quantity constraints on expenditure or supply are binding. It is not as elegant as the Walras' Law, referring not to actual excess demands (which are not everywhere well defined) but to their well-defined counterpart. Nevertheless, it serves a similar function in emphasizing the role of scarcity in proving the existence of general equilibrium. We saw this in Theorem 7.1 and we will see it again in Theorem 11.1.

## 10.5 Bibliographic note

The definition of household income as the value of endowment plus the share of firm profits appears in Arrow and Debreu (1954) and Debreu (1959).

## Exercises

10.1 In the economy with excess demand function $Z(p)$, the market for good $k$ is said to clear at prices $p \in P$ if $Z(p) \leq 0$, with $p_k = 0$ for $k$ such that $Z_k(p) < 0$. Recall the statement of the classic Walras Law For all $p \in P, p \cdot Z(p) = 0$.
A common interpretation of Walras' Law is: At prices $p \in P$, if there is market clearing in all markets but one (that is, in $N - 1$ markets) then the remaining ($Nth$) market clears as well. Explain and demonstrate the validity of the common interpretation.

10.2 Prove Theorem 10.1.

10.3 Prove Lemma 10.1. You will find the monotonicity assumption C.IV useful.