7

General equilibrium of the market economy with an excess demand function

7.1 Existence of equilibrium

In this chapter we will consider the existence of general equilibrium of an economy where demands $\tilde{D}^i(\cdot)$ and supplies $\tilde{S}^j(\cdot)$ come from bounded opportunity sets, $\tilde{B}^i(\cdot)$ and $\mathcal{Y}^j$, and are point valued. From Chapters 4 and 5 we know that a sufficient condition for point-valuedness is strict convexity of tastes and technologies, P.V and C.VII. As noted in Chapter 6, homogeneity of degree zero of $\tilde{D}^i(\cdot)$ and $\tilde{S}^j(\cdot)$ in p means that we may, without loss of generality, restrict the price space to be the unit simplex in $\mathbf{R}^N$

$$P = \left\{ p \mid p \in \mathbf{R}^N, p_k \geq 0, k = 1 \ldots, N, \sum_{k=1}^{N} p_k = 1 \right\}.$$

From Chapter 6, the market excess demand function is defined

$$\tilde{Z}(p) = \sum_{i \in H} \tilde{D}^i(\cdot) - \sum_{j \in F} \tilde{S}^j(\cdot) - r.$$

We are now in a position to define the general equilibrium of the market economy.

Definition $p^0 \in P$ is said to be an equilibrium price vector if $\tilde{Z}(p^0) \leq 0$ (the inequality holds coordinatewise) with $p^0_k = 0$ for k such that $\tilde{Z}_k(p^0) < 0$.

That is, an equilibrium is characterized by market clearing for all goods except perhaps free goods that may be in excess supply in equilibrium. To find sufficient conditions and to prove the existence of a general equilibrium, we have to focus on the excess demand function, $\tilde{Z}(p)$, $\tilde{Z} : P \to \mathbf{R}^N$. We have the following observations on $\tilde{Z}(p)$:

Weak Walras' Law (Theorem 6.2): For all $p \in P$, $p \cdot \tilde{Z}(p) \leq 0$. For p such

that $p \cdot \tilde{Z}(p) < 0$, there is $k = 1, 2, \ldots, N$ so that $\tilde{Z}_k(p) > 0$, under assumptions C.I–C.V, C.VII, and C.VIII, P.I–P.III, P.V, and P.VI.
Continuity: $\tilde{Z}(p)$ is a continuous function, assuming P.II, P.III, P.V, P.VI, C.I–C.V, and C.VII, C.VIII (Theorems 4.1, 5.2, and 6.1).

In addition, recall

Theorem 2.10 Brouwer Fixed-Point Theorem: Let S be an N-simplex and let $f : S \to S$, where f is continuous. Then there is $x^* \in S$ so that $f(x^*) = x^*$.

Our approach to proving the existence of general equilibrium follows the plan used in Chapter 1. We establish sufficient conditions so that excess demand is a continuous function of prices and fulfills the Weak Walras's Law (Theorem 6.2). The rest of the proof involves the mathematics of an economic story. Suppose the Walrasian auctioneer starts out with an arbitrary possible price vector (chosen at random, crié au hasard, in Walras's phrase) and then adjusts prices in response to the excess demand function $\tilde{Z}(p)$. He raises the price of goods, k, in excess demand, $\tilde{Z}_k(p) > 0$, and reduces the price of goods, k, in excess supply, $\tilde{Z}_k(p) < 0$. He performs this price adjustment as a continuous function of excess demands and supplies while staying on the price simplex. Then the price adjustment function $T(p)$ is a continuous mapping from the price simplex into itself. From the Brouwer Fixed-Point Theorem (Theorem 2.10), there is a fixed point p^0 of the price adjustment function, so that $T(p^0) = p^0$. Using the Weak Walras' Law we can then show that p^0 is not merely a fixed point of the price adjustment function, but it is a general equilibrium as well.

Theorem 7.1 [1] Assume P.II, P.III, P.V, P.VI, C.I–C.V, C.VII, and C.VIII. There is $p^* \in P$ so that p^* is an equilibrium.

Proof We formulate a price adjustment function, $T : P \to P$. Define $T(p)$ in the following fashion for each coordinate $k = 1, 2, 3, \ldots, N$:

$$T_k(p) \equiv \frac{p_k + \max[0, \tilde{Z}_k(p)]}{1 + \sum_{n=1}^{N} \max[0, \tilde{Z}_n(p)]} = \frac{p_k + \max[0, \tilde{Z}_k(p)]}{\sum_{n=1}^{N} \{p_n + \max[0, \tilde{Z}_n(p)]\}}.$$

The price adjustment function T raises the relative price of goods in excess demand and reduces that of goods in excess supply while keeping the price vector on the simplex. The price adjustment function here is specified dif-

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ferently — in this more general setting — from the one used in Chapter 1, to assure that the denominator (trivially) be positive.

By Lemma 6.1, $\tilde{Z}(p)$ is a continuous function. Then $T(p)$ is a continuous function from the simplex into itself since continuity is preserved under the operations of max, addition, and division by a positive-valued continuous function. An illustration of the notion of a continuous function from P into P is presented in Figure 7.1. By the Brouwer Fixed-Point Theorem there is $p^* \in P$ so that $T(p^*) = p^*$. But then for all $k = 1, \ldots, N$,

$$T_k(p^*) = p_k^* = \frac{p_k^* + \max[0, \tilde{Z}_k(p^*)]}{1 + \sum_{n=1}^{N} \max[0, \tilde{Z}_n(p^*)]}.$$

Thus, either Case 1 or Case 2 below applies.

CASE 1 $p_k^* = 0 = \max[0, \tilde{Z}_k(p^*)]$. Hence $\tilde{Z}_k(p^*) \leq 0$.

CASE 2

$$p_k^* = \frac{p_k^* + \max[0, \tilde{Z}_k(p^*)]}{1 + \sum_{n=1}^{N} \max[0, \tilde{Z}_n(p^*)]} > 0.$$

To avoid repeated tedious notation, let

$$0 < \alpha = \frac{1}{1 + \sum_{n=1}^{N} \max[0, \tilde{Z}_n(p^*)]} \leq 1.$$

We have

$$T_k(p^*) = p_k^* = \alpha(p_k^* + \max[0, \tilde{Z}_k(p^*)])$$

$$p_k^* = \alpha p_k^* + \alpha \max[0, \tilde{Z}_k(p^*)]$$

or

$$(1 - \alpha) p_k^* = \alpha \max[0, \tilde{Z}_k(p^*)].$$

Multiplying through by $\tilde{Z}_k(p^*)$, we get

$$(1 - \alpha) p_k^* \tilde{Z}_k(p^*) = \alpha(\max[0, \tilde{Z}_k(p^*)]) \tilde{Z}_k(p^*) \qquad (*).$$

We can restate the Weak Walras' Law as

$$0 \geq p^* \cdot \tilde{Z}(p^*) = \sum_{k \in \text{Case1}} p_k^* \tilde{Z}_k(p^*) + \sum_{k \in \text{Case2}} p_k^* \tilde{Z}_k(p^*)$$

$$= 0 + \sum_{k \in \text{Case2}} p_k^* \tilde{Z}_k(p^*)$$
$$= \sum_{k \in \text{Case2}} p_k^* \tilde{Z}_k(p^*).$$

Multiplying through by $(1-\alpha)$, and substituting per (*) we get

$$0 \geq (1-\alpha) \sum_{k \in \text{Case2}} p_k^* \tilde{Z}_k(p^*) = \alpha \sum_{k \in \text{Case2}} (\max[0, \tilde{Z}_k(p^*)])\tilde{Z}_k(p^*).$$

Then (strictly) positive entries in the sum on the right-hand-side will occur for $k \in$ Case 2 so that $\tilde{Z}_k(p^*) > 0$. Thus

$$0 \geq \sum_{k \in \text{Case2}, \tilde{Z}_k(p^*)>0} [\tilde{Z}_k(p^*)]^2.$$

But this means that $\tilde{Z}_k(p^*) \leq 0$, for all k in Case 2. However, then, there is no k, either in Case 1 or 2, so that $\tilde{Z}_k(p^*) > 0$. From the Weak Walras' Law it follows that $p^* \cdot \tilde{Z}(p^*) = 0$. Hence for k so that $\tilde{Z}_k(p^*) < 0$, it follows that $p_k^* = 0$. This completes the proof. QED

It is useful to remark on the character of the equilibrium in Theorem 7.1. We formalize this as

Lemma 7.1 Assume P.II, P.III, P.V, P.VI, C.I–C.V, CVII, and C.VIII. Let p^* be an equilibrium. Then $|\tilde{D}^i(p^*)| < c$, where c is the bound on the Euclidean length of demand, $\tilde{D}^i(p^*)$. Further, in equilibrium, Walras' Law holds as an equality: $p^* \cdot \tilde{Z}(p^*) = 0$.

Proof Since $\tilde{Z}(p^*) \leq 0$ (coordinatewise), we know that
$\sum_{i \in H} \tilde{D}^i(p^*) \leq \sum_{j \in F} \tilde{S}^j(p^*) + \sum_{i \in H} r^i$,
where the inequality holds coordinatewise. However, that implies that the aggregate consumption $\sum_{i \in H} \tilde{D}^i(p^*)$ is attainable, so for each household i, $|\tilde{D}^i(p^*)| < c$, where c is the bound on demand, $\tilde{D}^i(\cdot)$.

We have for all p, $p \cdot \tilde{Z}(p) \leq 0$. In equilibrium, at p^*, we have $\tilde{Z}(p^*) \leq 0$ (coordinatewise) with $p_k^* = 0$ for k so that $\tilde{Z}_k(p^*) < 0$. Therefore $p^* \cdot \tilde{Z}(p^*) = 0$. QED

We have now demonstrated the existence of equilibrium in the strictly convex bounded economy. Note how boundedness has entered the argument above. The technology sets of the firms, $\mathcal{Y}^j$, were assumed to be bounded. It follows that the technology set for the economy as a whole, $\mathcal{Y}$, is also bounded. In defining the opportunity sets of the households $\tilde{B}^i(\cdot)$ we constrained the household to choose a consumption plan in a bounded set (the closed ball of radius c) that contained the attainable points as a proper subset.

In the next several chapters, Part C, we will weaken the assumptions of boundedness used here. We consider there firms that recognize that their technology includes the possibility that with unbounded inputs they could produce unbounded outputs – prices will then nevertheless guide them to bounded inputs and outputs. We would like to weaken the boundedness restriction on household choice. Households should feel free to choose arbitrarily large consumption plans. In equilibrium, prices will lead the households to bounded plans, but it should be prices, not definitions, that do so. Indeed, according to Lemma 7.1, prices have already done that job in the equilibrium developed above. The typical household equilibrium consumption plan does not face a binding constraint on the Euclidean length of the consumption vector in equilibrium. That is, $|\tilde{D}^i(p^*)| < c$ (a strict inequality). We take advantage of this observation in Part C. We will demonstrate that putting that much faith in the price system is indeed confidence well placed.

7.2 Bibliographic note

The major mathematical insight of modern general equilibrium theory is the importance of the fixed-point theorem in proving the existence of equilibrium. It appears first in Arrow and Debreu (1954) and McKenzie (1954). The mapping used here appears in Varian (1992).

Exercises

7.1 Consider a two-commodity economy with an excess demand function $\tilde{Z}(p)$. $p \in P = \{p \mid p \in \mathbf{R}^2, p \geq 0, p_1 + p_2 = 1\}$. Let (p) be continuous, bounded, and fulfill Walras' Law as an equality $(p \cdot \tilde{Z}(p) = 0)$, and assume $\tilde{Z}_1(0,1) > 0$, $\tilde{Z}_2(1,0) > 0$. Without using the Brouwer Fixed-Point Theorem, show that the economy has an equilibrium. (Note: You may find the Intermediate Value Theorem useful.)

We use the following model (paralleling the model of Chapters 4–7) in Exercises 7.2 and 7.3. There is thought to be a finite set of firms denoted F. Each firm j is characterized by a production technology set $Y^j \subset \mathbf{R}^N$. There is a finite set of households H. Each household i is characterized by an endowment vector $r^i \in \mathbf{R}^N_+$, ownership share of

firm j, α^{ij}, and preferences depicted equivalently by the continuous monotone quasi-order $\succeq_i$ or by a utility function $u^i(\cdot)$, defined on a possible consumption set $X^i \subseteq \mathbf{R}^N$. In a private ownership economy, i's income is characterized as$M^i(p) = p \cdot r^i + \sum_{j \in F} \alpha^{ij} p \cdot y^{0j}$, where y^{0j} is firm j's profit-maximizing production plan. We will generally assume (except as noted in the questions) the standard conditions:
for households: income sufficient to keep consumption interior to the possible consumption set, weak monotonicity, continuity, and strict convexity of preferences;
for firms: continuity (closedness) and strict convexity of technology.
We use the following definition.

Definition $\{p^0, x^{0i}, y^{0j}\}$, $p^0 \in \mathbf{R}^N_+$, $i \in H$, $j \in F$, $x^{0i} \in \mathbf{R}^N$, $y^{0j} \in \mathbf{R}^N$ is said to be a competitive equilibrium if

(i) $y^{0j} \in Y^j$ and $p^0 \cdot y^{0j} \geq p^0 \cdot y$ for all $y \in Y^j$, for all $j \in F$,

(ii) $x^{0i} \in X^i$, $p^0 \cdot x^{0i} \leq M^i(p^0)$ and $x^{0i}_i x$ for all $x \in X^i$ with $p^0 \cdot x \leq M^i(p^0)$ for all $i \in H$, and

(iii)$0 \geq \sum_{i \in H} x^{0i} - \sum_{j \in F} y^{0i} - \sum_{i \in H} r^i$ with $p^0_k = 0$ for coordinates k so that the strict inequality holds.

7.2 Consider the general competitive equilibrium of a production economy with redistributive taxation of income from endowment. Half of each household's income from endowment (based on actual endowment, not net sales) is taxed away. The proceeds of the tax are then distributed equally to all households. We thus have

$$M^i(p) = p \cdot (.5r^i) + \sum_{j \in F} \alpha^{ij} p \cdot y^j + T,$$

where T is the transfer of tax revenues to the household,

$$T = (1/\#H) \sum_{h \in H} p \cdot (.5r^h).$$

Does there exist a competitive equilibrium in the economy with redistributive income taxation? Explain.

7.3 Consider the general competitive equilibrium of a production economy with excise taxation. In addition to the prices of goods $p \in \mathbf{R}^N_+$, there is a vector of excise taxes $\tau \in \mathbf{R}^N_+$. Proceeds of the tax are then distributed to households as a lump sum. Household income then is

$$M^i(p) = p \cdot r^i + \sum_{j \in F} \alpha^{ij} p \cdot y^j + T,$$

where T is the transfer of tax revenues to the household. The household budget constraint is

$$(p + \tau) \cdot x^i \leq M^i(p).$$

The transfer to the typical household, T, is then characterized as

$$T = (1/\#H) \sum_{h \in H} \tau \cdot x^h.$$

Does there exist a competitive equilibrium in the economy with excise taxation? Explain.

7.4 In an economy with an excess demand function $Z(\cdot), Z : P \to \mathbf{R}^N$, we usually define an equilibrium price vector as $p \in P$ so that $Z(p) \leq 0$ (where 0 is the zero vector, and the weak inequality holds coordinatewise), with $p_k = 0$ for any good k so that $Z_k(p) < 0$.

Some authors use an alternate definition:
p^* so that $Z(p^*) \leq 0$. That is, p^* is a Walrasian equilibrium if there is no good for which there is a positive excess demand.

The alternate definition imposes no requirement that $p^*_k = 0$ for k so that $Z_k(p^*) < 0$.

(i) Show that under this definition of equilibrium there may be excess supplies at positive prices in equilibrium.
(ii) What is the behavior of the market price adjustment process (Walrasian auctioneer) with excess supplies implied by this concept of equilibrium?
(iii) Discuss. Is this a desirable concept of equilibrium?

7.5 The usual U-shaped cost curve model of undergraduate economics includes a small nonconvexity (diminishing marginal cost at low output levels). This is a violation of our usual convexity assumptions on production (P.I or P.V). Consider the general equilibrium of an economy displaying U-shaped cost curves. It is possible that a general equilibrium exists despite the small violation of convexity. After all, P.I and P.V are sufficient, not necessary, conditions. Draw a diagram or give an example (partial equilibrium is acceptable). Explain. Nevertheless, it is also possible that an equilibrium fail to exist in this setting. Draw a diagram or give an example. Explain.

7.6 In Chapter 7 we used the mapping $T : P \to P$ as a price adjustment function whose fixed points are competitive equilibria. Consider instead using the mapping $Q : P \to P$, where the ith coordinate mapping of Q is

$$Q_i(p) = \frac{\max[0, p_i + p_i \tilde{Z}_i(p)]}{\sum_{j=1}^{N} \max[0, p_j + p_j \tilde{Z}_j(p)]}.$$

Assume that Walras' Law holds as an equality, that is, that $p\cdot(p) = 0$.

(i) Show that every competitive equilibrium price vector p^0 is a fixed point of Q.

(ii) Show that every vertex of the price simplex P is also a fixed point of Q.

(iii) Under suitably chosen sufficient conditions on the economy, $Q(\cdot)$ can be shown to have a fixed point, $p^* = Q(p^*)$. Does this prove that the economy – under those sufficient conditions – has a competitive equilibrium?

7.7 Consider the following definition: $\{p^0, x^{0i}, y^{0j}\}$, $p^0 \in \mathbf{R}_+^N$, $i \in H$, $j \in F$, is said to be a competitive equilibrium if

(i) $y^{0j} \in Y^j$ and $p^0 \cdot y^{0j} \geq p^0 \cdot y$ for all $y \in Y^j$, for all $j \in F$,

(ii) $x^{0i} \in X^i$, $p^0 \cdot x^{0i} \leq M^i(p^0) = p^0 \cdot r^i + \sum_{j \in F} \alpha^{ij} p^0 \cdot y^{0j}$ and $x_i^{0i} x$ for all $x \in X^i$ with $p^0 \cdot x \leq M^i(p^0)$ for all $i \in H$, and

(iii) $0 \geq \sum_{i \in H} x^{0i} - \sum_{j \in F} y^{0i} - \sum_{i \in H} r^i$ with $p_k^0 = 0$ for coordinates k so that the strict inequality holds.

(a) The concept of competitive equilibrium is supposed to reflect decentralization of economic behavior. Explain how this definition embodies the concept of decentralization.

(b) The concept of competitive equilibrium is supposed to reflect market clearing. Explain how this definition includes market clearing.

7.8 The style of analysis we have been using is known as "axiomatic," involving precisely stated assumptions, detailed modeling, and logically derived conclusions. What are the strengths and weaknesses of this approach?

7.9 A two-person, two-commodity, pure exchange (no production) economy is known as an Edgeworth box (discussed more fully in Section 1.3 – you should not need to consult this material). Use the model of Chapters 4 to 7 to demonstrate the existence of equilibrium in an Edgeworth box. Present the following argument:

(1) Set $\mathcal{Y}^j \equiv \{0\}$ for all $j \in F$, where 0 is the zero vector in $\mathbf{R}^N$. Explain why this represents the case of a pure exchange economy. Explain why the usual assumptions on production are fulfilled

by this choice of $\mathcal{Y}^j$.

(2) Define an equilibrium in this setting.

(3) Show that Theorem 7.1 applies and ensures the existence of equilibrium. State any additional assumptions you need.