PROBLEM SET 2 — Suggested Answers

It's OK to work together on problem sets.

All notation not otherwise defined is taken from Starr's General Equilibrium Theory *If you need to make additional assumptions to answer a question, that's OK. Do state the additional assumptions clearly.*

1. *Taken from Starr's* General Equilibrium Theory, *problem 1.5* Consider the following example of supply and demand relations between two markets. There are two goods, denoted 1 and 2, with prices p_1 and p_2, supply functions $S_1(p_1, p_2)$ and $S_2(p_1, p_2)$, and demand functions $D_1(p_1, p_2)$ and $D_2(p_1, p_2)$. These are specified by the expressions

$$S_1(p_1, p_2) = 3p_1; \qquad D_1(p_1, p_2) = 8 - 4p_2 - p_1; p_2 \leq 2$$

and

$$S_2(p_1, p_2) = 5p_2; \qquad D_2(p_1, p_2) = 24 - 6p_1 - p_2; p_1 \leq 4.$$

The market for good 1 is said to be in equilibrium at prices (p_1^o, p_2^o) where $S_1(p_1^o, p_2^o) = D_1(p_1^o, p_2^o)$. The market for good 2 is said to be in equilibrium at prices (p_1', p_2') where $S_2(p_1', p_2') = D_2(p_1', p_2')$. Demonstrate that each market has an equilibrium when the other's price is fixed. Show that, nevertheless, no pair of prices exists for the two markets at which they are both in equilibrium. Does this supply-demand system provide a counterexample to Theorem 1.2, on the existence of general equilibrium prices? Explain fully.

Suggested Answer : To arrange equilibrium in the market for good 1, we have
$S_1(p_1, p_2) = 3p_1 = D_1(p_1, p_2) = 8 - 4p_2 - p_1$
$3p_1 = 8 - 4p_2 - p_1$
$4p_1 = 8 - 4p_2$
$p_1 = 2 - p_2$, so there are many partial equilibrium solutions, e. g. $p_1 = p_2 = 1$.
To arrange equilibrium in the market for good 2, we have
$S_2(p_1, p_2) = 5p_2 = D_2(p_1, p_2) = 24 - 6p_1 - p_2$
$5p_2 = 24 - 6p_1 - p_2$
$6p_2 = 24 - 6p_1$
$p_2 = 4 - p_1$
$4 - p_2 = p_1$ so there are many partial equilibrium solutions, e.g. $p_1 = 1, p_2 = 3$.

For a general equilibrium, we must satisfy partial equilibrium in each of the two markets separately, so we have
$p_1 = 2 - p_2$ and $4 - p_2 = p_1$
$2 - p_2 = 4 - p_2$

$2 = 4$, an impossibility.

Hence there is no (p_1^*, p_2^*) simultaneously providing market clearing in both the market for good 1 and good 2.

However, this example is not a counterexample to our existence of general equilibrium theorems. Though the demand system is continuous in the range specified, it will not generally fulfill the Walras Law. Hence there is no counterexample implied.

2. *Taken from Starr's General Equilibrium Theory, problem 2.18*

The Brouwer Fixed-Point Theorem can be stated in the following way:

Let $S \subset \mathbf{R}^N$ be compact and convex. Let $f : S \to S$ be a continuous function. Then there is $x^* \in S$ so that $f(x^*) = x^*$.

Show how a fixed point would fail to exist when the assumptions of the Brouwer Fixed-Point theorem are not fulfilled, as specified in the following cases:

i. Suppose S is not convex. Let $S = [1, 2] \cup [3, 4]; S \subset \mathbf{R}$. That is, S is the union of two disjoint closed intervals in $\mathbf{R}$. Find continuous $f : S \to S$ so that there is no fixed point x^* fulfilling the theorem.

ii. Suppose f is not continuous. Let $S = [1, 4]; S \subset \mathbf{R}$. Let

$$f(x) = \begin{cases} 4 - x \text{ for } x < 2, \\ x - 1 \text{ for } x \geq 2. \end{cases}$$

Show that although $f : S \to S$ there is no fixed point of f in S.

iii. Suppose S is not compact. Let $S = \mathbf{R}$ and $f(x) = x + 1$. Note that $f : S \to S$ and f is continuous. Show that there is no fixed point of f in S.

Suggested Answer :(i) Suppose S is not convex. Let $S = [1, 2] \cup [3, 4]; S \subset R$. That is, S is the union of two disjoint closed intervals in R. Find continuous $f : S \to S$ so that there is no fixed point x* fulfilling the theorem.

Let f(x) = 5 - x. $f : [1, 2] \to [3, 4]; f : [3, 4] \to [1, 2]$. $f : S \to S$ and f continuous but there is no fixed point.

(ii) Suppose f is not continuous. Let $S = [1, 4]; S \subset R$. Let $f(x) = 4 - x$ for $x < 2$, $f(x) = x - 1$ for $x \geq 2$. Show that $f : S \to S$ but there is no fixed point of f in S.

For $1 \leq x < 2,$ we have $3 \geq f(x) > 2,$

$2 \leq x \leq 4,$ we have $1 \geq f(x) \leq 3, f(x) \neq x$. So $f : S \to S$, but there is no fixed point.

(iii) Suppose S is not compact. Let $S = R, f(x) = x + 1$. Show that $f : S \to S$ but there is no fixed point of f in S.

To have a fixed point would require that $f(x) = x + 1 = x$, or equivalently that $0 = 1$. This is impossible so there is no fixed point.

3. *Taken from Starr's General Equilibrium Theory Problem 2.19*

Recall the Intermediate Value Theorem:

Let $[a, b]$ be a closed interval in $\mathbf{R}$ and h a continuous real-valued function on $[a, b]$ so that $h(a) < h(b)$. Then for any real k so that $h(a) < k < h(b)$ there is $x \in [a, b]$ so that $h(x) = k$.

Recall the Brouwer Fixed-Point Theorem:

Let $S \subset R^N$ be compact and convex. Let $f : S \to S$ be a continuous function. Then there is $x^* \in S$ so that $f(x^*) = x^*$.

Consider the special case $S = [0, 1]$, the unit interval in $\mathbf{R}$, and let f be a continuous function from S into itself. Using the Intermediate Value Theorem, prove the Brouwer Fixed-Point Theorem for this case. You may find the function $g(x) = x - f(x)$ useful.

Suggested Answer : Define $g(x) = x - f(x)$. Then $g(0) \leq 0$ and $g(1) \geq 0$. Then by the Intermediate Value Theorem there is $x^* \in [0, 1]$ so that $g(x^*) = 0$. But then $g(x^*) = 0 = x^* - f(x^*)$ so $x^* = f(x^*)$ and x^* is the required fixed point.