Economics 113

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# Lecture Notes, Lecture 26

## Salvaging Majority Rule: Borda Count, Single Peaked Preferences and the Median Voter Theorem

Arrow Possibility Theorem's Four Conditions: Non-Dictatorship, Weak Pareto Principle, Independence of Irrelevant Alternatives, Unrestricted Domain. Omit any one of them and there is a rational group decision-making process that fulfills the remaining three.

Omit non-dictatorship: Appoint someone with transitive preferences as king.

Omit weak Pareto principle: Write transitive preferences (unaffected by any voter's preferences) into the constitution.
Omit independence of irrelevant alternatives: Borda Count (weighted voting).
Omit unrestricted domain: use majority voting on pairwise alternatives for population with single-peaked preferences.

Arrow Possibility Theorem implies that majority rule or any similar decision-making mechanism on pairwise alternatives cannot generally lead to transitive group preferences.

Restriction on space of possible preferences --- purposely violate 'Unrestricted Domain'; limit the space of possible profiles. Single peaked preferences: Suppose all propositions to be decided can be linearly ordered, left to right. All voters agree on the left to right ordering. They disagree on their choices.

Everyone has his favorite point; but *chacun a son gout* --- the favorite point differs among voters. For each voter, as we move to the left of his favorite his utility goes down; as we move to the right of his favorite his utility goes down.

Let L be the "is to the left of" ordering. All voters agree on the L ordering. Arrange the propositions $a_1$, $a_2$, .... so that $a_1$ L $a_2$ L $a_3$ L $a_4$ .... , and so forth. For each voter $i \in H$, there is a favorite proposition $a^{*i}$. All propositions to the left of $a^{*i}$ are inferior --- according to i's preferences --- and the farther to the left the worse they get. All propositions to the right of $a^{*i}$ are inferior to $a^{*i}$, and the farther to the right they get, the worse they are. Thus, for propositions u, v, w, x,

$$u \; L \; v \; L \; a^{*i} \; L \; w \; L \; x$$

implies $a^{*i}\, P_i\, v\, P_i\, u$, and $a^{*i}\, P_i\, w\, P_i\, x$. This situation describes "single-peaked preferences."

Arrange the favorite points of all agents $i \in H$, $a^{*i}$, in the left to right ordering. Assume (for convenience) an odd number of voters to avoid ties. Find the proposition $\mathscr{A}$ in the middle of this left to right array (so that half but one of others' favorites are to the left, half but one to the right). Then $\mathscr{A}$ is said to be the median preference point. It will command a majority vote against any alternative.

<u>Theorem 1</u> (Duncan Black): If preferences are single-peaked, then majority voting on pairwise alternatives yields transitive group decisions.

<u>Theorem 2</u> (Median voter theorem, Duncan Black): Let $\mathscr{A}$ be a median preference point. Then there is a majority (non-minority) of voters favoring $\mathscr{A}$ over any alternative, a'. (The favorite of the median voter is undominated in majority rule).

Proof of theorem 2: By inspection.

Proof of theorem 1: This requires some work. What do we want to show? Let P be the majority rule preference relation. Without loss of generality, let A P B, B P C, and let preferences be single peaked. Then we must show that A P C.

Consider (an exhaustive list of) six special cases:

1. A L B L C
2. B L C L A
3. C L A L B
4. C L B L A (equivalent argument to case 1)
5. A L C L B (equivalent argument to case 2)

6. B L A L C (equivalent argument to case 3)

Describe each household's preferences by a utility function $u^i()$. A household votes in favor of x over y when $u^i(x) > u^i(y)$. We will ignore ties.

Case 1: Consider those households $i \in H$, so that $u^i(A) > u^i(B)$. These households constitute a majority since A P B. But with the ordering of case 1, they must all have $u^i(B) > u^i(C)$ (otherwise they would fail single peakedness; they'd have two peaks). Hence we have A P C, as claimed.

Case 2: We claim case 2 is an empty set under A P B, B P C and single peakedness.

We have that a majority of voters has $u^i(B) > u^i(C)$. With the Case 2 ordering and single peakedness that means that a majority has $u^i(B) > u^i(A)$. Then we cannot have A P B, so case 2 cannot occur under the hypothesis.

Case 3: Really requires some work. We break H into four subgroups:

Households $i \in H$, so that:

Group I : $u^i(A) > u^i(B)$ ; $u^i( B) > u^i(C)$ . Transitivity of $u^i($ $)$ implies
$u^i(A) > u^i(C)$ .

Group II : $u^i(A) > u^i(B)$ ; $u^i( B) < u^i(C)$

Group III : $u^i(A) < u^i(B)$ ; $u^i(B) > u^i(C)$ . Single peakedness and the case 3 ordering implies that $u^i(A) > u^i(C)$ for Group III

Group IV : $u^i(A) < u^i(B)$ ; $u^i(B) < u^i(C)$. Single peakedness and the case 3 ordering implies that group IV is the empty set.

A P B implies $I \cup II$ constitutes a majority.

B P C implies $I \cup III$ constitutes a majority. Note preferences on A versus C in I and III. Then $I \cup III$ constitutes a majority for $u^i(A) > u^i(C)$,
so A P C as required. QED