Supplement to Lecture Notes of November 9: Production with unbounded technology, revised November 5, 2010

We restate for the technologies Y^j the assumptions P.I–P.III on production technologies introduced in Chapter 11 for the technology sets $\mathcal{Y}^j$:

(P.I) Y^j is convex for each $j \in F$.
(P.II) $0 \in Y^j$ for each $j \in F$.
(P.III) Y^j is closed for each $j \in F$.

The aggregate technology set is $Y = \sum_{j \in F} Y^j$.

Boundedness of the attainable set

(P.IV)(a) if $y \in Y$ and $y \neq 0$, then $y_k < 0$ for some k.
(b) if $y \in Y$ and $y \neq 0$, then $-y \notin Y$.

P.IV is not an assumption about the individual firms; it treats the production sector of the whole economy.
$r \in \mathbf{R}^N_+$ = vector of total initial resources or endowments.

Definition Let $y \in Y$. Then y is said to be attainable if $y + r \geq 0$ (the inequality holds co-ordinatewise).

In an attainable production plan $y \in Y$, $y = y^1 + y^2 + \ldots + y^{\#F}$, we have $y + r \geq 0$. But an individual firm's part of this plan, y^j , need not satisfy $y^j + r \geq 0$. Thus

Definition We say that $y^j \in Y^j$ is attainable in Y^j if there exists a $y^k \in Y^k$ for each of the firms $k \in F$, $k \neq j$, such that $y^j + \sum_{k \in F, k \neq j} y^k$ is attainable.

Lemma 15.1 Assume P.II and P.IV. Let $y = \sum_{j \in F} y^j,\ y^j \in Y^j$ for all $j \in F,\ y \in Y,\ y = \mathbf{0}$. Then $y^j = \mathbf{0}$ for all $j \in F$.

Theorem 15.1 For each $j \in F$, under P.I, P.II, P.III, and P.IV, the set of vectors attainable in Y^j is bounded.

Proof We will use a proof by contradiction. Suppose contrary to the theorem that the set of vectors attainable in $Y^{j'}$ is not bounded for some $j' \in F$. Then, for each $j \in F$, there exists a sequence $\{y^{\nu j}\} \subset Y^j, \nu = 1, 2, 3, \ldots$, such that:

(1) $|y^{\nu j'}| \to +\infty$, for some $j' \in F$,
(2) $y^{\nu j} \in Y^j$, for all $j \in F$, and
(3) $y^{\nu} = \sum_{j \in F} y^{\nu j}$ is attainable; that is, $y^{\nu} + r \geq 0$.

We show that this contradicts P.IV. Recall P.II, $0 \in Y^j$, for all j. Let $\mu^{\nu} = \max_{j \in F} |y^{\nu j}|$. For ν large, $\mu^{\nu} \geq 1$. By (1) we have $\mu^{\nu} \to +\infty$. Consider the sequence $\tilde{y}^{\nu j} \equiv \frac{1}{\mu^{\nu}} y^{\nu j} = \frac{1}{\mu^{\nu}} y^{\nu j} + (1 - \frac{1}{\mu^{\nu}})0$. By P.I, $\tilde{y}^{\nu j} \in Y^j$. Let $\tilde{y}^{\nu} = \frac{1}{\mu^{\nu}} y^{\nu} = \sum_{j \in F} \tilde{y}^{\nu j}$. By (3) and P.I we have

(4) $\tilde{y}^{\nu} + \frac{1}{\mu^{\nu}} r \geq 0$.

The sequences $\tilde{y}^{\nu j}$ and $\tilde{y}^{\nu}$ are bounded ($\tilde{y}^{\nu}$ as the finite sum of vectors of length less than or equal to 1). Without loss of generality, take corresponding convergent subsequences so that $\tilde{y}^{\nu} \to \tilde{y}^{\circ}$ and $\tilde{y}^{\nu j} \to \tilde{y}^{\circ j}$ for each j, and $\sum_j \tilde{y}^{\nu j} \to \sum_j \tilde{y}^{\circ j} = \tilde{y}^{\circ}$. Of course, $\frac{1}{\mu^{\nu}} r \to 0$. Taking the limit of (4), we have

$$\tilde{y}^{\circ} + 0 = \sum_{j \in F} \tilde{y}^{\circ j} + 0 \geq 0 \text{ (the inequality holds co-ordinatewise)} .$$

By P.III, $\tilde{y}^{\circ j} \in Y^j$, so $\sum_{j \in F} \tilde{y}^{\circ j} = \tilde{y}^{\circ} \in Y$. But, by P.IV(a), we have that $\sum_{j \in F} \tilde{y}^{\circ j} = 0$. Lemma 15.1 says then that $\tilde{y}^{\circ j} = \mathbf{0}$ for all j, so $|\tilde{y}^{\circ j}| \neq 1$.

The contradiction proves the theorem. QED

Theorem 15.2 Under P.I–P.IV, the set of attainable vectors in Y is compact, that is, closed and bounded.

Proof We will demonstrate the result in two steps.

Boundedness: $y \in Y$ attainable implies $y = \sum_{j \in F} y^j$ where $y^j \in Y^j$ is attainable in Y^j. However, by Theorem 15.1, the set of such y^j is bounded for each j. Attainable y then is the sum of a finite number ($\#F$) of vectors, y^j, each taken from a bounded subset of Y^j, so the set of attainable y in Y is also bounded.

Closedness: Consider the sequence $y^{\nu} \in Y$, y^{ν} attainable, $\nu = 1, 2, 3, \ldots$. We have $y^{\nu} + r \geq 0$. Suppose $y^{\nu} \to y^{\circ}$. We wish to show that $y^{\circ} \in Y$ and that y° is attainable. We write the sequence as $y^{\nu} = y^{\nu 1} + y^{\nu 2} + \ldots + y^{\nu j} + \ldots + y^{\nu \# F}$, where $y^{\nu j} \in Y^j$, $y^{\nu j}$ attainable in Y^j for all $j \in F$.

Since the attainable points in Y^j constitute a bounded set (by Theorem 15.1), without loss of generality, we can find corresponding convergent subsequences $y^{\nu}, y^{\nu 1}, y^{\nu 2}, \ldots, y^{\nu j}, \ldots, y^{\nu \# F}$ so that for all $j \in F$ we have $y^{\nu j} \to y^{\circ j} \in Y^j$, by P.III. We have then $y^{\circ} = y^{\circ 1} + y^{\circ 2} + \ldots + y^{\circ j} + \ldots + y^{\circ \# F}$ and $y^{\circ} + r \geq 0$. Hence, $y^{\circ} \in Y$ and y° is attainable. QED