FINAL EXAMINATION — SUGGESTED ANSWERS

This exam is take-home, open-book, open-notes. You may consult any published source (cite your references). Other people are closed. The exam you turn in should be your own personal work. Do not discuss with classmates, friends, professors (except with Ross or Troy — who promise to be clueless), until the examination is collected.

The exam is due by 5:00 PM, Tuesday, December 6, 2011. Submit your exam to Prof. Starr's lockbox on the ground floor of the Economics Building (the hallway near Econ 100).

Answer all 5 (five) questions. .

All notation not otherwise defined is taken from Starr's General Equilibrium Theory, second edition. If you need to make additional assumptions to answer a question, that's OK. Do state the additional assumptions clearly.

1. This question focuses on Figure 5.D.3 and the surrounding text taken from Mas-Colell, Whinston, and Green. The figure represents the case of U-shaped cost curves, often presented in undergraduate microeconomics courses.

   The usual representation of a production function with two inputs, x and y, would be $q = f(x, y), \frac{\partial f}{\partial x} > 0, \frac{\partial f}{\partial y} > 0, \frac{\partial^2 f}{\partial x^2} \leq 0, \frac{\partial^2 f}{\partial y^2} \leq 0, \frac{\partial^2 f}{\partial x \partial y} > 0$. Is this representation consistent with the situation depicted in Figure 5.D.3? Explain fully. Is the situation depicted in Figure 5.D.3 consistent with the existence of general equilibrium in an Arrow-Debreu economy with production? Explain fully.

   Suggested Answer: The depiction is a U-shaped cost curve, characterized by limited but significant scale economies at low levels of output. Hence the characterization $\frac{\partial^2 f}{\partial x^2} \leq 0, \frac{\partial^2 f}{\partial y^2} \leq 0$, will not hold everywhere in the relevant range. There will be regions where $\frac{\partial^2 f}{\partial x^2} > 0, \frac{\partial^2 f}{\partial y^2} > 0$. Traditional sufficient conditions for existence of Arrow-Debreu general equilibrium typically include convexity of production technology, (P.I or P.V in Starr's *General Equilibrium Theory*). So this setting violates traditional sufficient conditions for existence of general equilibrium. However, we have not demonstrated that convexity is a necessary condition for existence of general competitive equilibrium, so existence of equilibrium is still possible, but it cannot be assured. {Extra credit answer: Since the scale economies in the U-shaped cost curve case are bounded, there will exist approximate general equilibrium prices and allocations, the level of approximation depending on the size of the scale economies and the size of the economy.}

2. The weak monotonicity property on household preferences can be stated in the following fashion. Let $\succeq_i$ represent household i's preference ordering, or equivalently let $u^i(\cdot)$ represent i's utility function on an N-dimensional consumption possibility set.

   (Weak Monotonicity) Let $x, y \in \mathbf{R}^N_+$ and $x >> y$ that is $x_n > y_n$ for all $n = 1, 2, \cdots, N$. Then $x \succ_i y$ or equivalently $u^i(x) > u^i(y)$.

   The First Fundamental Theorem of Welfare Economics can be stated in the following fashion.

   First Fundamental Theorem of Welfare Economics For each $i \in H$, assume Weak Monotonicity and that each household consumption set is $\mathbf{R}^N_+$. Let $\{p^\circ, w^{\circ i}, y^{\circ j}\}$ represent a competitive equilibrium, where $p^\circ \in \mathbf{R}^N_+$ is the equilibrium price vector,$w^{\circ i} \in \mathbf{R}^N_+$, for a household i, is the associated individual consumption bundle, and $y^{\circ j}$, for firm j, is the associated firm supply vector. Then $w^{\circ i}$ is Pareto efficient.

   Consider a two-person pure exchange economy (Edgeworth Box) made up of the following two households. The notation "$\min[xy, 16]$" means the minimum of xy and 16. Superscripts denote the household name — nothing in this problem is raised to a power.

	Household 1	Household 2
Endowment	$r^1 = (1, 9)$	$r^2 = (9, 1)$
Utility Function	$u^1(x, y) = xy$	$u^2(x, y) = \min[xy, 16]$

   (a) Household 2 has a maximum utility of 16; whenever household 2's holdings of x and y fulfill $xy > 16$, household 2 gets no additional satisfaction from additional consumption. Adopt the notation: (x^1, y^1) is household 1's consumption plan of x and y; (x^2, y^2) is household 2's consumption plan of x and y. Set $p = (.5, .5)$. This is a competitive equilibrium price vector with the consumption plan $(x^1, y^1) = (5, 5)$, $(x^2, y^2) = (5, 5)$. Show that this plan is Pareto inefficient.

   (b) Is this a counterexample to the First Fundamental Theorem of Welfare Economics? Explain.

Suggested Answer: The consumption plan is Pareto inefficient inasmuch as there is a Pareto preferable attainable consumption plan: $(x^1, y^1) = (6, 6)$, $(x^2, y^2) = (4, 4)$. It is not a counterexample to 1FTWE inasmuch as sufficient conditions for the theorem are not fulfilled; household 2's preferences are not weakly monotone, but satiable.

3. Consider a small economy, with two goods and three households. The two goods are denoted x, y. The households have identical preferences on $\mathbf{R}^2_+$ described by the utility function

$$u(x, y) = \sup[x, y]$$

where "sup" indicates the supremum or maximum of the two arguments. These tastes could be characterized by the household saying

"I like x and y equally well, and more is definitely better. But they are redundant. When there's more x, I use the x and discard the y. And when there's more y, it's y that I use and discard the x."

The households have identical endowments of (10, 10).

(i) Demonstrate that there is no competitive equilibrium in this economy [Hint: Show that price vector $(\frac{1}{2}+\varepsilon, \frac{1}{2}-\varepsilon)$, $\varepsilon > 0$, cannot be an equilibrium; similarly for $(\frac{1}{2}-\varepsilon, \frac{1}{2}+\varepsilon)$; and finally $(\frac{1}{2}, \frac{1}{2})$. That pretty well takes care of it.]

(ii) The standard results for an Arrow-Debreu general equilibrium model include proofs of existence of general equilibrium. That result apparently fails in the example above. Explain. How can this happen? Is the example above a counterexample, demonstrating that the usual existence of general equilibrium results are invalid? Does the example above fulfill the usual sufficient conditions for existence of general equilibrium in an Arrow-Debreu model?

Suggested Answer: **(i)** Consider price vector $(\frac{1}{2}+\varepsilon, \frac{1}{2}-\varepsilon)$, $\varepsilon > 0$. In this case, all demand is for y (the less expensive good) and the excess demand vector $Z = (-30, 30) \neq (0, 0)$. Hence not an equilibrium.

Consider price vector $(\frac{1}{2}-\varepsilon, \frac{1}{2}+\varepsilon)$. Then all demand is for x, the less expensive good and the excess demand vector is $Z = (30, -30) \neq (0, 0)$, and hence not an equilibrium.

Finally consider the price vector $(\frac{1}{2}, \frac{1}{2})$. Then for each of three households, i, excess demand is $Z^i = (-10, 10)$ **or** $Z^i = (10, -10)$. Summing over the three households total excess demand is then $Z = (-30, 30)$, or $(-10, 10)$, or $(10, -10)$, or $(30, -30)$. There is no market clearing price vector.

(ii) The example is valid, but it is not a counterexample to the usual existence of general equilibrium theorems. Their sufficient conditions include convexity of preferences. The utility function presented here does not fulfill convexity. The households here are indifferent between $(20, 0)$ and $(0, 20)$ but either of those is superior to a convex combination of the two. Hence the usual existence theorems do not apply, and this example, though valid in itself, is not a counterexample to the usual existence of general equilibrium results.

4. Consider an Arrow-Debreu economy under uncertainty with a full set of contingent commodity markets over two periods. Period 1 is certain. Uncertainty is on the state of the world in period 2. The market takes place prior to any real economic activity. Let there be n commodities available in period 1 and in each contingency of period 2. Let S denote the set of possible states of the world in period 2, with typical element s. Let π^{is} be household i's subjective probability of state $s \in S$ in period 2. Let $v^i(x^s)$ denote the utility household i receives in state $s \in S$ from contingent commodity bundle x^s. Let x be an $n \times (1 + \#S)$ -dimensional vector denoting a portfolio of x^1 (with certainty) in period 1, and contingent commodity bundles x^s (each n-dimensional but varying with s) for period 2. Suppose the households $i \in H$, are expected utility maximizers without time discounting, so that their utility functions can be described as

$$u^i(x) = v^i(x^1) + \sum_{s \in S} \pi^{is} v^i(x^s).$$

Production in the economy is characterized by firms $j \in F$ with technology sets $Y^j \in \mathbf{R}^{n \times (1+\#S)}$.

(i) Let a competitive equilibrium price and allocation be established at the market date. The $n \times (1+\#S)$ -dimensional vector of period 1 prices and period 2 contingent commodity prices is denoted p. Each firm j seeks an $n \times (1+\#S)$ -dimensional production plan $y^j \in Y^j$ to maximize the value of $p \cdot y^j$. How does the firm take account of uncertainty? Does each firm formulate a probability distribution on future events in order to maximize expected discounted profits? Explain.

(ii) When period 2 arrives, uncertainty is resolved and state $s^* \in S$ prevails. Markets for spot goods are open for trade in period 2-state s^* goods. Some households will have correctly placed a high subjective probability on s*, others will be disappointed that they placed a much lower — or zero — subjective probability on the state that actually occurs. What trades will take place between those with correct predictions and the others who under-predicted? Explain.

Suggested Answer: **(i)** The firms need no forecast of the future, risk aversion, or subjective probability. The firm plans state-contingent production that it knows it can fulfill. The firm knows its state-contingent technology with certainty and knows the prevailing contingent commodity prices. Firm j does just what it does in a certainty setting. It chooses $y^j \in Y^j$ to maximize the value of $p \cdot y^j$. The firm does not deal directly with uncertainty. Attitudes to risk and subjective probabilities come from the household demand side of the economy and are embodied in prices.

(ii) There will be no trade in period 2. The allocation — within period 2, state s^* — is Pareto efficient. Denote the contingent commodity price of good n, date 2, state s, as p_n^{2s}. *Ex ante* all households h and i set

$$MRS^i_{n,m} = \frac{\pi^{is} v^i_n(x^s)}{\pi^{is} v^i_m(x^s)} = \frac{p_n^{2s}}{p_m^{2s}} = \frac{\pi^{hs} v^h_n(x^s)}{\pi^{hs} v^h_m(x^s)} = MRS^h_{n,m},$$

for all $s \in S$. Then it follows, *ex post*, that

$$MRS^{i,s^*}_{n,m} = \frac{v^i_n(x^{s^*})}{v^i_m(x^{s^*})} = \frac{v^h_n(x^{s^*})}{v^h_m(x^{s^*})} = MRS^{h,s^*}_{n,m},$$

which is a first-order condition for Pareto efficiency. The allocation within date 2, state s^* is Pareto efficient. There will be no additional trade.

22.7 Consider a pure exchange economy becoming large through Q-fold replication. Consider an example where there are two commodities, x and y, and two trader types, 1 and 2.

Type 1 is characterized as having utility function

$$u^1(x,y) = xy, \text{and endowment}$$
$$r^1 = (99,1).$$

Type 2 is characterized as having utility function

$$\begin{aligned} u^2(x,y) &= xy, \text{and endowment} \\ r^2 &= (1,99). \end{aligned}$$

1. Show that the following allocation, a^1 to type 1 and a^2 to type 2, is in the core for all levels of replication Q: a$^1 = a^2 = (50, 50)$

2. Show that the following allocation, a^1 to type 1 and a^2 to type 2, is in the core for the original economy with one of each type, and is not in the core for an economy with $Q \geq 2$:

$$\begin{aligned} a^1 &= (90, 90); \\ a^2 &= (10, 10). \end{aligned}$$

Define a new concept, the *equi-core*, as the set of allocations unblocked by *equal-weighted* coalitions. A coalition S in economy $Q - H$ will be said to equal-weighted , if it contains **the same number** of individuals of each type represented in the coalition. For example, a coalition of five households each of types 1 and 2 is equal-weighted; a coalition of five of type 1, and four of type 2 is not equal-weighted; a coalition of three of type 2 and zero of type 1 is equal-weighted.

You may assume without proof that the equi-core retains two properties of the core: inclusion of the competitive equilibrium (Theorem 21.1), and the equal treatment property (Theorem 22.1). Further, you may assume that any equal-weighted blocking coalition maintains the equal treatment property in its blocking allocation.

(c) Show that the following allocation, discussed in part (b), a^1 to type 1 and a^2 to type 2, is in the equi-core for the original economy with one of each type, and is still in the equi-core for an economy with $Q \geq 2$:

$$\begin{aligned} a^1 &= (90, 90); \\ a^2 &= (10, 10). \end{aligned}$$

(d) Discuss the examples of parts (b) and (c). What do they indicate about the process of core convergence in Theorem 22.2 ?

Suggested Answer:

(a) Theorem 21.1 says that the competitive equilibrium allocation is always in the core. $p = (.5, .5)$ is a CE price vector with CE allocation $a^1 = a^2 = (50, 50)$. Hence that allocation is in the core for Q=1. But in a replica economy, the CE prices and allocations stay the same for all Q, so this allocation is always in the core.

(b) For $Q = 1$ the only available coalitions are the whole and singletons. $a^1 = (90, 90); a^2 = (10, 10)$ is Pareto efficient so the whole will not block it; it is individually rational since $90 \times 90 = 8100 > 99 \times 1 = 99$ and $10 \times 10 = 100 > 99 = 99 \times 1$. Hence the allocation is in the core for $Q = 1$.

For $Q = 2$ form the coalition of two of type 2 and one of type 1. Form the blocking allocation $a^1 = (130, 70); a^{2-1} = (35, 15); a^{2-2} = (34, 16)$. The allocation improves the utility for each member of the coalition, so it blocks. Hence $a^1 = (90, 90); a^2 = (10, 10)$ is not in the core for $Q = 2$.

(c) For $Q = 1$ the core and the equi-core coincide so $a^1 = (90, 90); a^2 = (10, 10)$ remains in the equi-core for $Q = 1$. For $Q = 2$, suppose there is a blocking coalition S. S cannot be a singleton — it would have blocked at $Q = 1$. S cannot be two of one type —– that type could then have blocked at $Q = 1$. S cannot be two elements consisting of one of each type —- then S would have blocked at $Q = 1$. Hence S must be a four-element coalition, two of each type. Denote the blocking allocation $a^1 = (x^1, y^1); a^2 = (x^2, y^2)$. The blocking allocation must be Pareto improving for the coalition; without loss of generality, suppose it is strictly improving for type 2. Then $u^1(a^1) \geq 90 \times 90; u^2(a^2) > 10 \times 10$ and $2a^1 + 2a^2 \leq [2 \times (99, 1) + 2 \times (1, 99)]$. But this leads to a contradiction. If S can block in $Q = 2$, then the coalition of the whole could have blocked the same allocation in $Q = 1$. Hence the allocation $a^1 = (90, 90); a^2 = (10, 10)$ is not blocked in the equi-core of $Q = 2$.

(d) The inability of the equi-core to shrink as the economy becomes large illustrates the dynamic of shrinkage in the core. It is the ability to form increasingly asymmetric (not equal-weighted) coalitions as the replication becomes larger that allows the core to converge.