

Lecture Notes, January 29, 2014

Bargaining and equilibrium: The core of a market economy

Set $X^i = \mathbf{R}_+^N$, all i .

Each $i \in H$ has an endowment $r^i \in \mathbf{R}_+^N$ and a preference quasi-ordering \succeq_i defined on \mathbf{R}_+^N .

An allocation is an assignment of $x^i \in \mathbf{R}_+^N$ for each $i \in H$. A typical allocation, $x^i \in \mathbf{R}_+^N$ for each $i \in H$, will be denoted $\{x^i, i \in H\}$. An allocation, $\{x^i, i \in H\}$, is feasible if $\sum_{i \in H} x^i \leq \sum_{i \in H} r^i$, where the inequality holds coordinatewise.

We assume preferences fulfill weak monotonicity (C.IV*), continuity (C.V), and strict convexity (C.VI(SC)).

The core of a pure exchange economy

Definition A *coalition* is any subset $S \subseteq H$. Note that every individual comprises a (singleton) coalition.

Definition An allocation $\{x^i, i \in H\}$ is *blocked* by $S \subseteq H$ if there is a coalition $S \subseteq H$ and an assignment $\{y^i, i \in S\}$ so that:

- (i) $\sum_{i \in S} y^i \leq \sum_{i \in S} r^i$ (where the inequality holds coordinatewise),
- (ii) $y^i \succeq_i x^i$, for all $i \in S$, and
- (iii) $y^h \succ_h x^h$, for some $h \in S$

Definition The *core* of the economy is the set of feasible allocations that are not blocked by any coalition $S \subseteq H$.

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- Any allocation in the core must be individually rational. That is, if $\{x^i, i \in H\}$ is a core allocation then we must have $x^i \succeq_h r^i$, for all $i \in H$.
 - Any allocation in the core must be Pareto efficient.
- (i) The competitive equilibrium is always in the core (Theorem 21.1).

Theorems 22.2 and 22.3 say that

- (ii) For a large economy, the set of competitive equilibria and the core are virtually identical. All core allocations are (nearly) competitive equilibria.

The competitive equilibrium allocation is in the core

Definition $p \in \mathbf{R}_+^N$, $p \neq 0$, $x^i \in \mathbf{R}_+^N$, for each $i \in H$, constitutes a competitive equilibrium if

- (i) $p \cdot x^i \leq p \cdot r^i$, for each $i \in H$,
- (ii) $x^i \succeq_i y$, for all $y \in R_+^N$, such that $p \cdot y \leq p \cdot r^i$, and
- (iii) $\sum_{i \in H} x^i \leq \sum_{i \in H} r^i$ (the inequality holds coordinatewise) with $p_k = 0$ for any $k = 1, 2, \dots, N$ so that the strict inequality holds.

Theorem 21.1 Let the economy fulfill C.II, C.IV*, C.VI(SC) and let $X^i = \mathbf{R}_+^N$. Let $p, x^i, i \in H$, be a competitive equilibrium. Then $\{x^i, i \in H\}$ is in the core of the economy.

Proof We will present a proof by contradiction. Suppose the theorem were false. Then there would be a blocking coalition

$S \subseteq H$ and a blocking assignment $y^i, i \in S$. We have

$$\begin{aligned} \sum_{i \in S} y^i &\leq \sum_{i \in S} r^i \text{ (attainability, the inequality holds coordinatewise)} \\ y^i &\succeq_i x^i, && \text{for all } i \in S, \text{ and} \\ y^h &\succ_h x^h, && \text{some } h \in S. \end{aligned}$$

But x^i is a competitive equilibrium allocation. That is, for all $i \in H$, $p \cdot x^i = p \cdot r^i$ (recalling Lemma 17.1), and $x^i \succeq_i y$, for all $y \in R_+^N$ such that $p \cdot y \leq p \cdot r^i$.

Note that $\sum_{i \in S} p \cdot x^i = \sum_{i \in S} p \cdot r^i$. Then for all $i \in S$, $p \cdot y^i \geq p \cdot r^i$. That is, x^i represents i 's most desirable consumption subject to budget constraint. y^i is at least as good under preferences \succeq_i fulfilling C.II, C.IV*, C.VI(SC), (local non-satiation). Therefore, y^i must be at least as expensive. Furthermore, for h , we must have $p \cdot y^h > p \cdot r^h$. Therefore, we have

$$\sum_{i \in S} p \cdot y^i > \sum_{i \in S} p \cdot r^i.$$

Note that this is a strict inequality. However, for coalitional feasibility we must have

$$\sum_{i \in S} y^i \leq \sum_{i \in S} r^i.$$

But since $p \geq 0$, $p \neq 0$, we have $\sum_{i \in S} p \cdot y^i \leq \sum_{i \in S} p \cdot r^i$. This is a contradiction. The allocation $\{y^i, i \in S\}$ cannot simultaneously be smaller or equal to the sum of endowments r^i coordinatewise and be more expensive at prices $p, p \geq 0$. The contradiction proves the theorem. QED

Convergence of the core of a large economy

Replication; a large economy

In replication, the economy keeps cloning itself.

duplicate to triplicate, \dots , to Q -tuplicate, and so on, the set of core allocations keeps getting smaller, although it always includes the set of competitive equilibria (per Theorem 21.1).

Q -fold replica economy, denoted Q - H . $Q = 1, 2, \dots$

$\#H \times Q$ agents.

Q agents with preferences \succeq_1 and endowment r^1 ,

Q agents with preferences \succeq_2 and endowment r^2, \dots , and Q agents with preferences $\succeq_{\#H}$ and endowment $r^{\#H}$. Each household $i \in H$ now corresponds to a household type. There are Q individual households of type i in the replica economy Q - H .

Competitive equilibrium prices in the original H economy will be equilibrium prices of the Q - H economy. Household i 's competitive equilibrium allocation x^i in the original H economy will be a competitive equilibrium allocation to all type i households in the Q - H replica economy. Agents in the Q - H replica economy will be denoted by their type and a serial number. Thus, the agent denoted i, q will be the q th agent of type i , for each $i \in H, q = 1, 2, \dots, Q$.

Equal treatment

Theorem 22.1 (Equal treatment in the core) Assume C.IV, C.V, and C.VI(SC). Let $\{x^{i,q}, i \in H, q = 1, \dots, Q\}$ be in the core of Q - H , the Q -fold replica of economy H . Then for each $i, x^{i,q}$ is the same for all q . That is, $x^{i,q} = x^{i,q'}$ for each $i \in H, q \neq q'$.

Proof of Theorem 22.1 Recall that the core allocation must be feasible. That is,

$$\sum_{i \in H} \sum_{q=1}^Q x^{i,q} \leq \sum_{i \in H} \sum_{q=1}^Q r^i.$$

Equivalently,

$$\frac{1}{Q} \sum_{i \in H} \sum_{q=1}^Q x^{i,q} \leq \sum_{i \in H} r^i.$$

Suppose the theorem to be false. Consider a type i so that $x^{i,q} \neq x^{i,q'}$. For each type i , we can rank the consumptions attributed to type i according to \succeq_i .

For each i , let x^{i*} denote the least preferred of the core allocations to type i , $x^{i,q}$, $q = 1, \dots, Q$. For some types i , all individuals of the type will have the same consumption and x^{i*} will be this expression. For those in which the consumption differs, x^{i*} will be the least desirable of the consumptions of the type. We now form a coalition consisting of one member of each type: the individual from each type carrying the worst core allocation, x^{i*} .

Consider the average core allocation to type i , to be denoted \bar{x}^i .

$$\bar{x}^i = \frac{1}{Q} \sum_{q=1}^Q x^{i,q}.$$

We have, by strict convexity of preferences (C.VI(SC)),

$$\bar{x}^i = \frac{1}{Q} \sum_{q=1}^Q x^{i,q} \succ_i x^{i*} \text{ for those types } i \text{ so that } x^{i,q} \text{ are not identical,}$$

and

$$x^{i,q} = \bar{x}^i = \frac{1}{Q} \sum_{q=1}^Q x^{i,q} \sim_i x^{i*} \text{ for those types } i \text{ so that } x^{i,q} \text{ are identical.}$$

From feasibility, above, we have that

$$\sum_{i \in H} \bar{x}^i = \sum_{i \in H} \frac{1}{Q} \sum_{q=1}^Q x^{i,q} = \frac{1}{Q} \sum_{i \in H} \sum_{q=1}^Q x^{i,q} \leq \sum_{i \in H} r^i.$$

In other words, a coalition composed of one of each type (the worst off of each) can achieve the allocation \bar{x}^i . However, for each agent in the coalition, $\bar{x}^i \succeq_i x^{i*}$ for all i and $\bar{x}^i \succ_i x^{i*}$ for

some i . Therefore, the coalition of the worst off individual of each type blocks the allocation $x^{i,q}$. The contradiction proves the theorem. QED

$\text{Core}(Q) = \{x^i, i \in H\}$ where $x^{i,q} = x^i, q = 1, 2, \dots, Q$, and the allocation $x^{i,q}$ is unblocked.

Core convergence in a large economy

As Q grows there are more blocking coalitions, and they are more varied. Any coalition that blocks an allocation in Q - H still blocks the allocation in $(Q + 1)$ - H , but there are new blocking coalitions and allocations newly blocked in $(Q + 1)$ - H .

Recall the Bounding Hyperplane Theorem:

Theorem 8.1, Bounding Hyperplane Theorem (Minkowski) Let K be convex, $K \subseteq \mathbf{R}^N$. There is a hyperplane H through z and bounding for K if z is not interior to K . That is, there is $p \in \mathbf{R}^N, p \neq 0$, so that for each $x \in K, p \cdot x \geq p \cdot z$.

Theorem 22.2 (Debreu-Scarf) Assume C.IV*, C.V, C.VI(SC), and let $X^i = \mathbf{R}_+^N$. Let $\{x^{oi}, i \in H\} \in \text{core}(Q)$ for all $Q = 1, 2, 3, 4, \dots$. Then $\{x^{oi}, i \in H\}$ is a competitive equilibrium allocation for Q - H , for all Q .

Proof TBA

QED