Lecture Notes for January 15, 2014

A market economy

Firms, profits, and household income

H , F, $\alpha^{ij} \in \mathbf{R}_+$, $\sum_{i\in H} \alpha^{ij} = 1$,

$$r \equiv \sum_{i \in H} r^i.$$

Theorem 13.1 Assume P.II, P.III, and P.VI. $\tilde{\pi}^j(p)$ is a well-defined continuous function of p for all $p \in \mathbf{R}^N_+, p \neq 0$. $\tilde{\pi}^j(p)$ is homogeneous of degree 1.

$\tilde{M}^i(p) = p \cdot r^i + \sum_{j\in F} \alpha^{ij} \tilde{\pi}^j(p)$.

$$P = \left\{ p \mid p \in \mathbf{R}^N, p_k \geq 0, k = 1 \ldots, N, \sum_{k=1}^{N} p_k = 1 \right\}.$$

Excess demand and Walras' Law

Definition The excess demand function at prices $p \in P$ is

$$\tilde{Z}(p) = \tilde{D}(p) - \tilde{S}(p) - r = \sum_{i \in H} \tilde{D}^i(p) - \sum_{j \in F} \tilde{S}^j(p) - \sum_{i \in H} r^i.$$

Lemma 13.1 Assume C.I–C.V, C.VI(SC), C.VII, P.II, P.III, P.V, and P.VI. The range of $\tilde{Z}(p)$ is bounded. $\tilde{Z}(p)$ is continuous and well defined for all $p \in P$.

Proof Apply Theorems 11.1, 12.2, 13.1. The finite sum of bounded sets is bounded. The finite sum of continuous functions is continuous. QED

Theorem 13.2 (Weak Walras' Law) Assume C.I–C.V, C.VI(SC),C.VII, P.II, P.III, P.V, and P.VI. For all $p \in P$, $p{\bullet}\tilde{Z}(p) \leq 0$. For p such that $p{\bullet}\tilde{Z}(p) < 0$, there is $k = 1, 2, \ldots, N$ so that $\tilde{Z}_k(p) > 0$.

Proof of Theorem 13.2 $p{\bullet}\tilde{D}^i(p) \leq \tilde{M}^i(p) = p{\bullet}r^i + \sum_{j\in F} \alpha^{ij}\tilde{\pi}^j(p)$. $\sum_{i\in H} \alpha^{ij} = 1$ for each $j \in F$.

$$\begin{aligned}
p{\bullet}\tilde{Z}(p) &= p{\bullet}\left[\sum_{i\in H} \tilde{D}^i(p) - \sum_{j\in F} \tilde{S}^j(p) - \sum_{i\in H} r^i\right] \\
&= p{\bullet}\sum_{i\in H} \tilde{D}^i(p) - p{\bullet}\sum_{j\in F} \tilde{S}^j(p) - p{\bullet}\sum_{i\in H} r^i \\
&= \sum_{i\in H} p\cdot\tilde{D}^i(p) - \sum_{j\in F} p\cdot\tilde{S}^j(p) - \sum_{i\in H} p\cdot r^i \\
&= \sum_{i\in H} p\cdot\tilde{D}^i(p) - \sum_{j\in F} \tilde{\pi}^j(p) - \sum_{i\in H} p\cdot r^i \\
&= \sum_{i\in H} p\cdot\tilde{D}^i(p) - \sum_{j\in F}\left[\sum_{i\in H} \alpha^{ij}\tilde{\pi}^j(p)\right] - \sum_{i\in H} p\cdot r^i \\
&= \sum_{i\in H} p\cdot\tilde{D}^i(p) - \sum_{i\in H}\left[\sum_{j\in F} \alpha^{ij}\tilde{\pi}^j(p)\right] - \sum_{i\in H} p\cdot r^i \\
&\quad \text{Note the change in the order of summation} \\
&= \sum_{i\in H} p\cdot\tilde{D}^i(p) - \sum_{i\in H}\left\{\left[\sum_{j\in F} \alpha^{ij}\tilde{\pi}^j(p)\right] + p\cdot r^i\right\} \\
&= \sum_{i\in H} p\cdot\tilde{D}^i(p) - \sum_{i\in H} \tilde{M}^i(p) \\
&= \sum_{i\in H}\left[p\cdot\tilde{D}^i(p) - \tilde{M}^i(p)\right] \leq 0.
\end{aligned}$$

since $p{\bullet}\tilde{D}^i(p) \leq \tilde{M}^i(p)$ This proves the weak inequality as required.

We now must demonstrate the positivity of some coordinate of $\tilde{Z}(p)$ when the strict inequality holds. Let $p{\cdot}\tilde{Z}(p){<}0$. Then $p{\cdot}\sum_{i\in H}\tilde{D}^i(p){<}p{\cdot}r{+}p{\cdot}\sum_{j\in F}\tilde{S}^j(p) = \sum_{i\in H}\tilde{M}^i(p)$, so for some $i'\in H$, $p{\cdot}\tilde{D}^{i'}(p) < \tilde{M}^{i'}(p)$. Now we apply Lemma 12.3. We must have $|\tilde{D}^{i'}(p)| = c$. Recall that c is chosen so that $|x| < c$ (a strict inequality) for all attainable x. But then $\tilde{D}^{i'}(p)$ is not attainable. For no $y \in \mathcal{Y}$ do we have $\tilde{D}^{i'}(p) \le y + r$. But for all $i \in H$, $\tilde{D}^i(p) \in \boldsymbol{R}^N_+$. So $\sum_{i\in H}\tilde{D}^i(p) \ge \tilde{D}^{i'}(p)$. Therefore, $\tilde{Z}_k(p) > 0$, for some $k = 1, 2, \ldots, N$. QED

General equilibrium of the market economy with an excess demand function

Existence of equilibrium

$$P = \left\{ p \mid p \in \mathbf{R}^N, p_k \ge 0, k = 1\ldots, N, \sum_{k=1}^{N} p_k = 1 \right\}.$$

$$\tilde{Z}(p) = \sum_{i\in H}\tilde{D}^i(\cdot) - \sum_{j\in F}\tilde{S}^j(\cdot) - r.$$

Definition $p^\circ \in P$ is said to be an equilibrium price vector if $\tilde{Z}(p^\circ) \le 0$ (the inequality holds coordinatewise) with $p^\circ_k = 0$ for k such that $\tilde{Z}_k(p^\circ) < 0$.

Weak Walras' Law (Theorem 13.2): For all $p \in P$, $p \cdot \tilde{Z}(p) \le 0$. For p such that $p \cdot \tilde{Z}(p) < 0$, there is $k = 1, 2, \ldots, N$ so that $\tilde{Z}_k(p) > 0$, under assumptions C.I–C.V, C.VI(SC), P.II, P.III, P.V, and P.VI.

Continuity: $\tilde{Z}(p)$ is a continuous function, assuming P.II, P.III, P.V, P.VI, C.I–C.V, C.VI(SC) and C.VII (Theorems 11.1, 12.2, and 13.1).

Theorem 9.3 Brouwer Fixed-Point Theorem: Let S be an N-simplex and let $f : S \to S$, where f is continuous. Then there is $x^* \in S$ so that $f(x^*) = x^*$.

Theorem 14.1 Assume P.II, P.III, P.V, P.VI, C.I–C.V, C.VI (SC), and C.VII. There is $p^* \in P$ so that p^* is an equilibrium.

Proof Let $T : P \to P$, where $T(p) = (T_1(p), T_2(p), \ldots, T_i(p), \ldots, T_N(p))$. $T_i(p)$ is the adjusted price of good i, adjusted by the auctioneer trying to bring supply and demand into balance. Let $\gamma^i > 0$. The adjustment process of the ith price can be represented as $T_i(p)$, defined as follows:

$$T_i(p) \equiv \frac{\max[0, p_i + \gamma^i \tilde{Z}_i(p)]}{\sum_{n=1}^{N} \max[0, p_n + \gamma^n \tilde{Z}_n(p)]}. \tag{14.1}$$

In order for T to be well defined, we must show that the denominator is nonzero, that is,

$$\sum_{n=1}^{N} \max[0, p_n + \gamma^n \tilde{Z}_n(p)] \neq 0. \tag{14.2}$$

In fact, we claim that $\sum_{n=1}^{N} \max[0, p_n + \gamma^n \tilde{Z}_n(p)] > 0$. Suppose not. Then for each n, $\max[0, p_n + \gamma^n \tilde{Z}_n(p)] = 0$. Then all goods k with $p_k > 0$ must have $\tilde{Z}_k(p) < 0$. So $p \cdot \tilde{Z}(p) < 0$. Then by the Weak Walras' Law, there is n so that $\tilde{Z}_n(p) > 0$. Thus $\sum_{n=1}^{N} \max[0, p_n + \gamma^n \tilde{Z}_n(p)] > 0$.

By Lemma 13.1, $\tilde{Z}(p)$ is a continuous function. Then $T(p)$ is a continuous function from the simplex into itself since continuity is preserved under the operations of max, addition, and division by a positive-valued continuous function.

By the Brouwer Fixed-Point Theorem there is $p^* \in P$ so that

$T(p^*) = p^*$. But then for all $k = 1, \ldots, N$,

$$T_i(p^*) \equiv \frac{\max[0, p_i^* + \gamma^i \tilde{Z}_i(p^*)]}{\sum_{n=1}^{N} \max[0, p_n^* + \gamma^n \tilde{Z}_n(p^*)]}. \tag{14.3}$$

We'll demonstrate that $\tilde{Z}_n(p^*) \leq 0$ all n.

Looking at the numerator in this expression, we can see that the equation will be fulfilled either by

$$p_k^* = 0 \qquad (Case\ \ 1) \tag{14.4}$$

or by

$$p_k^* = \frac{p_k^* + \gamma^k \tilde{Z}_k(p^*)}{\sum_{n=1}^{N} \max[0, p_n^* + \gamma^n \tilde{Z}_n(p^*)]} > 0 \qquad (Case\ \ 2). \tag{14.5}$$

CASE 1 $p_k^* = 0 = \max[0, p_k^* + \gamma^k \tilde{Z}_k(p^*)]$. Hence, $0 \geq p_k^* + \gamma^k \tilde{Z}_k(p^*) = \gamma^k \tilde{Z}_k(p^*)$ and $\tilde{Z}_k(p^*) \leq 0$. This is the case of free goods with market clearing or with excess supply in equilibrium.

CASE 2 To avoid repeated messy notation, let

$$\lambda = \frac{1}{\sum_{n=1}^{N} \max[0, p_n^* + \gamma^n \tilde{Z}_n(p^*)]} > 0 \tag{14.6}$$

so that $T_k(p^*) = \lambda(p_k^* + \gamma^k \tilde{Z}_k(p^*))$. We'll demonstrate that

$\tilde{Z}_n(p^*) \leq 0$, all n. Since p^* is the fixed point of T we have $p_k^* = \lambda(p_k^* + \gamma^k \tilde{Z}_k(p^*)) > 0$. This expression is true for all k with $p_k^* > 0$, and λ is the same for all k. Let's perform some algebra on this expression. We first combine terms in p_k^*:

$$(1 - \lambda) p_k^* = \lambda \gamma^k \tilde{Z}_k(p^*), \tag{14.7}$$

then multiply through by $\tilde{Z}_k(p^*)$ to get

$$(1-\lambda)p_k^*\tilde{Z}_k(p^*) = \lambda\gamma^k(\tilde{Z}_k(p^*))^2, \tag{14.8}$$

and now sum over all k in Case 2, obtaining

$$(1-\lambda)\sum_{k\in\text{Case2}} p_k^*\tilde{Z}_k(p^*) = \lambda \sum_{k\in\text{Case2}} \gamma^k(\tilde{Z}_k(p^*))^2. \tag{14.9}$$

The Weak Walras' Law says

$$0 \geq \sum_{k=1}^{N} p_k^*\tilde{Z}_k(p^*) = \sum_{k\in\text{Case1}} p_k^*\tilde{Z}_k(p^*) + \sum_{k\in\text{Case2}} p_k^*\tilde{Z}_k(p^*). \tag{14.10}$$

But for $k \in$ Case 1, $p_k^*\tilde{Z}_k(p^*) = 0$, and so

$$0 = \sum_{k\in\text{Case1}} p_k^*\tilde{Z}_k(p^*). \tag{14.11}$$

Therefore,

$$\sum_{k\in\text{Case2}} p_k^*\tilde{Z}_k(p^*) \leq 0. \tag{14.12}$$

There are two subcases, $\lambda \leq 1$, and $\lambda > 1$. In the case $\lambda \leq 1$, from (14.9) we have

$$0 \geq (1-\lambda)\cdot \sum_{k\in\text{Case2}} p_k^*\tilde{Z}_k(p^*) = \lambda \cdot \sum_{k\in\text{Case2}} \gamma^k(\tilde{Z}_k(p^*))^2. \tag{14.13}$$

The left-hand side $\leq$ 0. But the right-hand side is necessarily nonnegative. It can be zero only if $\tilde{Z}_k(p^*) = 0$ for all k such that $p_k^* > 0$ (k in Case 2). Thus, p^* is an equilibrium. This concludes the proof for the case $\lambda \leq 1$.

In the event $\lambda > 1$ equation (14.8) implies

$$(1-\lambda)p_k^*\tilde{Z}_k(p^*) \geq 0 \text{ for all } k \in \text{Case 2}.$$

Since $\lambda > 1$, this results in $\tilde{Z}_k(p^*) \leq 0$ for all $k \in$ Case 2. But there can be no $k \in$ Case 2 so that $\tilde{Z}_k(p^*) < 0$. If that were to occur, then $p^* \cdot \tilde{Z}(p^*) < 0$ and by the Weak Walras Law $\tilde{Z}_k(p^*) > 0$ for some $k \in$ Case 1 or Case 2, a contradiction. Hence in this subcase, we have $\tilde{Z}_k(p^*) = 0$ for all $k \in$ Case 2. This concludes the proof.

QED

Lemma 14.1 Assume P.II, P.III, P.V, P.VI, C.I–C.V, C.VI(SC), and C.VII. Let p^* be an equilibrium. Then for all $i \in H$, $|\tilde{D}^i(p^*)| < c$, where c is the bound on the Euclidean length of demand, $\tilde{D}^i(p^*)$. Further, in equilibrium, Walras' Law holds as an equality: $p^* \cdot \tilde{Z}(p^*) = 0$.

Proof Since $\tilde{Z}(p^*) \leq 0$ (coordinatewise), we know that

$\sum_{i \in H} \tilde{D}^i(p^*) \leq \sum_{j \in F} \tilde{S}^j(p^*) + \sum_{i \in H} r^i$,

where the inequality holds coordinatewise. However, that implies that the aggregate consumption $\sum_{i \in H} \tilde{D}^i(p^*)$ is attainable, so for each household i, $|\tilde{D}^i(p^*)| < c$, where c is the bound on demand, $\tilde{D}^i(\cdot)$.

We have for all p, $p \cdot \tilde{Z}(p) \leq 0$. In equilibrium, at p^*, we have $\tilde{Z}(p^*) \leq 0$ (co-ordinatewise) with $p_k^* = 0$ for k so that $\tilde{Z}_k(p^*) < 0$. Therefore $p^* \cdot \tilde{Z}(p^*) = 0$. QED