## Your Name:

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**Please answer all questions. Each of the six questions marked with a big number counts equally. Designate your answers clearly.**

**Short correct answers are sufficient and get full credit. Including irrelevant (though correct) information in an answer will not increase the score.**

**All notation not otherwise defined is taken from Starr's *General Equilibrium Theory: An Introduction (2nd edition).***

If you need additional space to answer a question, write "(over)" at the end of text on the first page and continue on the back of the page. If you still need additional space, use an additional sheet of paper, making sure that it has your name on it and is attached to your exam when submitted.

**This examination is open book, open notes (hard copy only). Calculators, cell phones, computers, iPads, etc., advice of classmates, are not allowed.**

# 1 Equal Treatment in the Core

C.V (Continuity) and C.VI(C) (Convexity; not strict convexity), imply

**Weak Convexity** Let $u^i(x) \geq u^i(y)$ and $0 \leq \alpha \leq 1$. Then $u^i(\alpha x + (1-\alpha)y) \geq u^i(y)$.

Consider the core of an economy growing large through replication under C.IV* (weak monotonicity), C.V and C.VI(C) but without C.VI(SC) (strict convexity). There may be some linear (flat) segments on indifference curves. The equal treatment property as stated by Debreu - Scarf (all households of the same type have identical core allocations) is no longer valid. Consider an alternative

**Weak Equal Treatment** Let $iq$ and $iq'$ be two distinct households of the same type $i$ and let $x^{iq}$ and $x^{iq'}$ be their core allocations in a Q-fold replica economy ($Q \geq 2$). Then $u^i(x^{iq}) = u^i(x^{iq'})$.

**CLAIM: The Weak Equal Treatment property holds in the core of a replica economy fulfilling C.IV* (weak monotonicity), C.V, C.VI(C).**

**Circle the correct answer and explain your reasoning.**

**TRUE** The claim is true. Start from a proposed core allocation failing the weak equal treatment property. Then form the same blocking coalition as in the Debreu-Scarf proof (Theorem 22.1); it will successfully block. Hence the claim can be demonstrated.

**FALSE** The claim is false. Even the *weak* equal treatment property and the possibility of forming a blocking coalition depend essentially on *strict* convexity.

**Explain:**

***Suggested Answer:*** **TRUE** The Debreu-Scarf proof (Theorem 22.1) forms a coalition of one of each type $i$ and allocates a consumption to that household of $\bar{x}^i$, the average consumption of the type. If there is unequal treatment (weak definition) then $u^i(\bar{x}^i) > u^i(x^{i^*})$ where $x^{i^*}$ is the least preferred allocation to type $i$ in the proposed core allocation. Hence the unequal treatment (weak definition) allocation is blocked.

## 2 Core Convergence in the Replica Economy

Consider a two-person pure exchange economy (Edgeworth Box) becoming large through replication. Start with the following two households. Superscripts denote the household name — nothing in this problem is raised to a power.

| | Household 1 | Household 2 |
|---|---|---|
| Endowment | $r^1 = (1, 99)$ | $r^2 = (99, 1)$ |
| Utility Function | $u^1(x, y) = x \times y$ | $u^2(x, y) = \sqrt{x \times y}$ |

**1.**Show that the following allocation is in the core of the original (unreplicated) economy: $(x^1, y^1) = (10, 10), (x^2, y^2) = (90, 90)$.

***Suggested Answer:*** $u^1(x^1, y^1) = 100 > 99 = u^1(r^1), u^2(x^2, y^2) = \sqrt{8100} > u^2(r^2)$ so the allocation is individually rational. It is unblocked since it is Pareto efficient, $\frac{u^1_x}{u^1_y} = \frac{y^1}{x^1} = \frac{\frac{\sqrt{y}^2}{\sqrt{x}^2}}{\frac{\sqrt{x}^2}{\sqrt{y}^2}} = \frac{u^2_x}{u^2_y}$, where the subscripts indicate partial derivatives.

**2.** Consider the same allocation (to type 1 and type 2), $(x^1, y^1) = (10, 10), (x^2, y^2) = (90, 90)$, in the 2-fold replica economy. Is the allocation still in the core? **Circle the correct answer below and explain.**

**YES** The allocation is individually rational and Pareto efficient. It remains in the core.

**NO** There is a blocking coalition consisting of two of type 1 and one of type 2. The original allocation to type is blocked.

**Explain:**

***Suggested Answer:*NO**

We form the blocking coalition allocation in the following way: To each of two type 1: (7,20); to one of type 2: (87, 179). Total resources allocated (101, 199). $u^1(7, 20) = 140 > 100; u^2(87, 179) = \sqrt{15723} > \sqrt{8100}$, so the coalition blocks.

## 3 Robinson Crusoe

Robinson Crusoe lives alone on an island. His possible consumption set is $X^i = \mathbf{R}^{\mathbf{2}}_{+}$, the nonnegative quadrant. The sole factor of production is his own labor $L$, supplied inelastically. He owns two firms, one making $x$ the other making $y$. The y-producing firm can make y in the quantity $(L^y)^{1.5}$ (the 1.5 power of $L^y$) where $L^y$ is the amount of labor devoted to y. The x-producing firm can make x in the quantity $(L^x)^{1.5}$ (the 1.5 power of $L^x$) where $L^x$ is the amount of labor devoted to x. Each output varies as the 1.5 power of the labor devoted to it. Restating:

$$x = (L^x)^{1.5}, \qquad\qquad y = (L^y)^{1.5}.$$

*All arithmetic values here are correct to one decimal place subject to roundoff.* There is a limit on the available labor, $L^x + L^y = 24$, $L^x, L^y \geq 0$. Robinson's utility function is $u(x, y) = inf[x, y]$, where "inf" indicates infimum or minimum. These are Leontieff preferences. The Pareto efficient allocation is $(x^\circ, y^\circ) = (41.6, 41.6) = (12^{1.5}, 12^{1.5})$ with labor allocated equally to production of x and y. Set $p_x = p_y = 1$ and the wage rate at 3.5. Then Robinson's income is $24 \times 3.5 = 83.2$ (with roundoff) and he can afford to purchase the efficient allocation, and production of x and y each generate a profit of zero. However, when he figures out the profits available from producing y, he discovers that setting $L^y = 24$ would yield an output of 118 yielding a net profit of 118 - 83.2 = 34.8 (with roundoff). The same thing happens when he figures out setting $L^x = 24$. And he still has sufficient income to buy all of the output, since he gets his wage plus the profits of the producing firm. Specializing production in either x or y is more profitable than diversifying between x and y. But producing (x, y) = (118, 0) or (0, 118) generates a utility of zero. Specializing is highly profitable and completely unsatisfactory.

- There is no competitive equilibrium in this model. Why? **Circle the response A, B, C, D or E below that best replies, then explain.**

**A.** C.VII (Adequacy of Income) may not be fulfilled, so there may be no competitive equilibrium.

**B.** Leontieff preferences fulfill C.VI(C) (convexity) but not C.VI(SC) (strict convexity). Hence there may be no general equilibrium price or allocation.

**C.** The scale economy in production is a violation of P.V (strict convexity) or P.I (convexity) so that competitive equilibrium may not exist.

**D.** Walras's Law fails so that the household cannot afford available supplies, leading to nonexistence of competitive equilibrium.

**E.** None of responses A, B, C, or D, is correct. See the explanation below.

**Explain:**

***Suggested Answer:*** **C.** The production technology, $x = (L^x)^{1.5}, \quad y = (L^y)^{1.5}$, reflects a scale economy. Doubling inputs more than doubles outputs. That is a nonconvexity, a violation of P.I and P.V. That is the reason that there is no equilibrium in this example.

## 4 Nonsatiation in 1FTWE

The First Fundamental Theorem of Welfare Economics and its proof are reproduced in Appendix 1. It requires local non-satiation, summarized as C.II ($X^i$ unbounded above), C.IV (non-satiation), and C.VI(C) (convex preferences) <u>or</u> C.IV* (weak monotonicity), $X^i = \mathbf{R}^N_+$.

Local non-satiation is the property that nearby to each possible consumption plan there is another strictly preferred.

In the proof in the appendix, the proof would fail without the property of local non-satiation. Which is the first numbered equation there that may be false without that assumption?

**Circle the number of the first equation that may fail, then explain below:**

**1 2 3 4 5 6 7**

**Explain:**

***Suggested Answer:*** 3 Equations 1 and 2 have little to do with non-satiation. Equation 3 says the household spends up to the limit of its budget. That is a consequence of non-satiation.

## 5 Existence of General Competitive Equilibrium

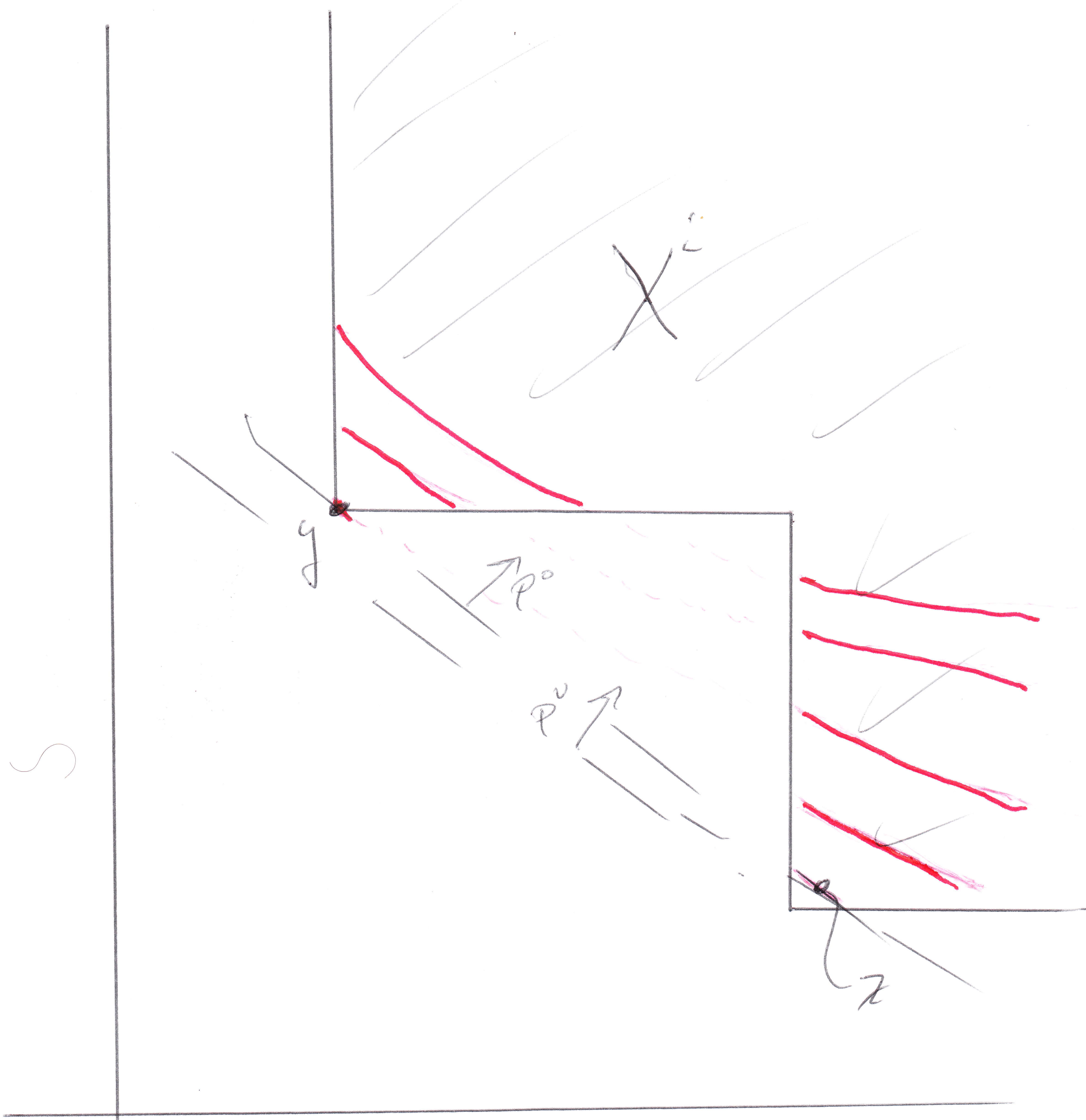
The statement and proof of existence of general equilibrium prices in an economy with bounded firm technology sets (P.VI), Theorem 14.1, is reproduced in Appendix 2. The presentation in the appendix is adapted from the treatment in *General Equilibrium Theory: An Introduction, 2nd edition*. You may not need to refer to the appendix.

**1.** How can the theorem fail if C.V (Continuity of household preferences $\succeq_i$ or utility $u^i(\cdot)$) is omitted? Explain.

***Suggested Answer:*** Without continuity of $u^i$ or $\succeq_i$ demand (and hence $\tilde{Z}(p)$) may be discontinuous so that the fixed point theorem need not apply. Discontinuous demand behavior with discontinuous $\succeq_i$ is illustrated in Starr's *General Equilibriium Theory 2nd edition* problem 14.20 example beta.

**2.** The theorem can fail if C.III ($X^i$ is convex) is omitted, since this can lead to discontinuity in $\tilde{D}^i(p)$ and therefore discontinuity in $\tilde{Z}(p)$ despite fulfillment of all other required assumptions. How can this occur? Explain; a diagram may help.

***Suggested Answer:*** Let $u^i(y) >> u^i(x)$ but the chord between $y$ and $x$ is not in $X^i$. Consider a sequence of prices $p^\nu \to p^o$ so that $p^\nu \cdot x = M^i(p^\nu) < p^\nu \cdot y$, but $p^o \cdot y = M^i(p^o)$. Then there can be a discontinuous jump in demand from $x$ to $y$ reflecting the nonconvexity of $X^i$.




5.2

# 6 Futures and contingent commodity markets over time

**1.**In a monetary economy, households save for retirement income by saving and investing money during their working lifetime to be spent later during retirement. Consider an Arrow-Debreu economy over time without uncertainty with a full set of futures markets. How is the saving and spending for retirement achieved in the Arrow-Debreu model? Explain.

***Suggested Answer:***Saving and securing distant future consumption takes the form of selling early-dated endowment and taking account of wealth, using the proceeds to buy later-dated consumption.

**2.** In an Arrow-Debreu model under uncertainty with a full set of contingent commodity markets, Starr's textbook states "the firm does not need a probability judgment to forecast which states are more likely nor does it need an attitude toward risk." How then does the firm make purchase and sale decisions in this uncertain environment? Explain.

***Suggested Answer:*** The firm — taking contingent commodity prices as parametrically given — consults its technology and maximizes the value of the firm, or indeed the firm's net contingent output, at prevailing contingent commodity prices. Since the contingent commodity prices are known independent of the probability of events, the firm can do this without making a probability judgment. Since the value of the firm can be fixed by these choices and rendered to shareholders at the market date, the firm need have no attitude toward risk.

## Appendix 1: First Fundamental Theorem of Welfare Economics

**Theorem 19.1**: For each $i \in H$ , assume C.II, C.IV, and C.VI(C) or assume C.IV*, $X^i = \mathbf{R}^N_+$. Let $p^\circ \in \mathbf{R}^N_+$ be a competitive equilibrium price vector of the economy. Let $w^{\circ i} \in X^i$, $i{\in}H$, be the associated individual consumption bundles, and let $y^{\circ j}$, $j \in F$, be the associated firm supply vectors. Then $w^{\circ i}$ is Pareto efficient.

**Proof**: $u^i(w^{\circ i}) > u^i(x)$, for all $x \in X^i$ so that $p^\circ \cdot x \le M^i(p^\circ)$, for all $i \in H$. This is a property of the equilibrium allocation. Consider an allocation $x^i$ that household $i \in H$ regards as more desirable than $w^{\circ i}$. If the allocation $x^i$ is preferable, it must also be more expensive. That is,

$$u^i(x^i) > u^i(w^{\circ i}) \text{ implies } p^\circ \cdot x^i > p^\circ \cdot w^{\circ i}. \quad \textbf{(1)}$$

Similarly, profit maximization in equilibrium implies that production plans more profitable than $y^{\circ j}$ at prices $p$ are not available in $Y^j$. $p^\circ \cdot y > p^\circ \cdot y^{\circ j}$ implies $y \notin Y^j$. Noting that markets clear at the equilibrium allocation, we have

$$\sum_{i\in H} w^{\circ i} \le \sum_{j \in F} y^{\circ j} + r. \quad \textbf{(2)}$$

Note that, for each household $i \in H$,

$$p^\circ \cdot w^{\circ i} = M^i(p^\circ) = p^\circ \cdot r^i + \sum_j \alpha^{ij}(p^\circ \cdot y^{\circ j}), \quad \textbf{(3)}$$

Summing over households,

$$\begin{aligned}
\sum_{i\in H} p^\circ \cdot w^{\circ i} = \sum_i M^i(p^\circ) &= \sum_i \left[ p^\circ \cdot r^i + \sum_j \alpha^{ij}(p^\circ \cdot y^{\circ j}) \right] \\
&= p^\circ \cdot \sum_i r^i + p^\circ \cdot \sum_i \sum_j \alpha^{ij} y^{\circ j} \\
&= p^\circ \cdot \sum_i r^i + p^\circ \cdot \sum_j \sum_i \alpha^{ij} y^{\circ j} \\
&= p^\circ \cdot r + p^\circ \cdot \sum_j y^{\circ j} \; \left( \text{since for each j, } \sum_i \alpha^{ij} = 1 \right). \quad \textbf{(4)}
\end{aligned}$$

Suppose, contrary to the theorem, there is an attainable Pareto preferable allocation $v^i \in X^i$, $i \in H$, so that $u^i(v^i) \ge u^i(w^{\circ i})$, for all $i$ with $u^h(v^h) > u^h(w^{\circ h})$ for some $h \in H$. The allocation $v^i$ must be more expensive than $w^{\circ i}$ for those households made better off and no less expensive for the others. Then we have

$$\sum_{i\in H} p^\circ \cdot v^i > \sum_{i \in H} p^\circ \cdot w^{\circ i} = \sum_{i\in H} M^i(p^\circ) = p^\circ \cdot r + p^\circ \cdot \sum_{j\in F} y^{\circ j}. \quad \textbf{(5)}$$

But if $v^i$ is attainable, then there is $y'^j \in Y^j$ for each $j \in F$, so that

$$\sum_{i\in H} v^i = \sum_{j\in F} y'^j + r. \quad \textbf{(6)}$$

But then, evaluating this production plan at the equilibrium prices, $p^\circ$, we have

$$p^\circ \cdot r + p^\circ \cdot \sum_{j \in F} y^{\circ j} < p^\circ \cdot \sum_{i \in H} v^i = p^\circ \cdot \sum_{j \in F} y'^j + p^\circ \cdot r. \qquad \textbf{(7)}$$

So $p^\circ \cdot \sum_{j \in F} y^{\circ j} < p^\circ \cdot \sum_{j \in F} y'^j$. Therefore, for some $j \in F$, $p^\circ \cdot y^{\circ j} < p^\circ \cdot y'^j$.

But $y^{\circ j}$ maximizes $p^\circ \cdot y$ for all $y \in Y^j$; there cannot be $y'^j \in Y^j$ so that $p \cdot y'^j > p \cdot y^{\circ j}$. Hence, $y'^j \notin Y^j$. The contradiction shows that $v^i$ is not attainable. **QED**

## Appendix 2: Existence of General Competitive Equilibrium in an Arrow - Debreu Economy

**Theorem 14.1:** Assume P.II, P.III, P.V, P.VI, C.I–C.V, C.VI (SC), and C.VII. There is $p^* \in P$ so that $p^*$ is an equilibrium.

**Proof:** Let $T : P \to P$, where $T(p) = (T_1(p), T_2(p), \ldots, T_i(p), \ldots, T_N(p))$. $T_i(p)$ is the adjusted price of good $i$, adjusted by the auctioneer trying to bring supply and demand into balance. Let $\gamma^i > 0$ . The adjustment process of the $i$th price can be represented as $T_i(p)$, defined as follows:

$$T_i(p) \equiv \frac{\max[0, p_i + \gamma^i \tilde{Z}_i(p)]}{\sum_{n=1}^{N} \max[0, p_n + \gamma^n \tilde{Z}_n(p)]}. \tag{8}$$

The function $T$ is a price adjustment function. It raises the relative price of goods in excess demand and reduces the price of goods in excess supply while keeping the price vector on the simplex. The expression $p_i + \gamma^i \tilde{Z}_i(p)$ represents the idea that prices of goods in excess demand should be raised and those in excess supply should be reduced. The operator $\max[0, \cdot]$ represents the idea that adjusted prices should be nonnegative. The fractional form of $T$ reminds us that after each price is adjusted individually, they are then readjusted proportionally to stay on the simplex. In order for $T$ to be well defined, we must show that the denominator is nonzero, that is,

$$\sum_{n=1}^{N} \max[0, p_n + \gamma^n \tilde{Z}_n(p)] \neq 0. \tag{9}$$

In fact, we claim that $\sum_{n=1}^{N} \max[0, p_n + \gamma^n \tilde{Z}_n(p)] > 0$. Suppose not. Then for each n, $\max[0, p_n + \gamma^n \tilde{Z}_n(p)] = 0$. Then all goods $k$ with $p_k > 0$ must have $\tilde{Z}_k(p) < 0$. So $p \cdot \tilde{Z}(p) < 0$. Then by the Weak Walras' Law, there is $n$ so that $\tilde{Z}_n(p) > 0$. Thus $\sum_{n=1}^{N} \max[0, p_n + \gamma^n \tilde{Z}_n(p)] > 0$.

By Lemma 13.1, $\tilde{Z}(p)$ is a continuous function. Then $T(p)$ is a continuous function from the simplex into itself since continuity is preserved under the operations of max, addition, and division by a positive-valued continuous function. An illustration of the notion of a continuous function from $P$ into $P$ is presented in Figure 14.1. By the Brouwer Fixed-Point Theorem there is $p^* \in P$ so that $T(p^*) = p^*$. But then for all $k = 1, \ldots, N$,

$$T_i(p^*) \equiv \frac{\max[0, p_i^* + \gamma^i \tilde{Z}_i(p^*)]}{\sum_{n=1}^{N} \max[0, p_n^* + \gamma^n \tilde{Z}_n(p^*)]}. \tag{10}$$

We'll demonstrate that $\tilde{Z}_n(p^*) \leq 0$ all $n$.

Looking at the numerator in this expression, we can see that the equation will be fulfilled either by

$$p_k^* = 0 \qquad (Case 1) \tag{11}$$

or by

$$p_k^* = \frac{p_k^* + \gamma^k \tilde{Z}_k(p^*)}{\sum_{n=1}^{N} \max[0, p_n^* + \gamma^n \tilde{Z}_n(p^*)]} > 0 \qquad (Case 2). \tag{12}$$

$p_k^* = 0 = \max[0, p_k^* + \gamma^k \tilde{Z}_k(p^*)]$. Hence, $0 \geq p_k^* + \gamma^k \tilde{Z}_k(p^*) = \gamma^k \tilde{Z}_k(p^*)$ and $\tilde{Z}_k(p^*) \leq 0$. This is the case of free goods with market clearing or with excess supply in equilibrium.

To avoid repeated messy notation, define

$$\lambda \equiv \frac{1}{\sum_{n=1}^{N} \max[0, p_n^* + \gamma^n \tilde{Z}_n(p^*)]} > 0 \tag{13}$$

so that $T_k(p^*) = \lambda(p_k^* + \gamma^k \tilde{Z}_k(p^*))$. We'll demonstrate that $\tilde{Z}_n(p^*) \leq 0$ all $n$ . Since $p^*$ is the fixed point of $T$ we have $p_k^* = \lambda(p_k^* + \gamma^k \tilde{Z}_k(p^*)) > 0$. This expression is true for all $k$ with $p_k^* > 0$, and $\lambda$ is the same for all $k$. Let's perform some algebra on this expression. We first combine terms in $p_k^*$:

$$(1-\lambda)p_k^* = \lambda \gamma^k \tilde{Z}_k(p^*), \tag{14}$$

then multiply through by $\tilde{Z}_k(p^*)$ to get

$$(1-\lambda)p_k^* \tilde{Z}_k(p^*) = \lambda \gamma^k (\tilde{Z}_k(p^*))^2, \tag{15}$$

and now sum over all $k$ in Case 2, obtaining

$$(1-\lambda) \sum_{k \in \text{Case2}} p_k^* \tilde{Z}_k(p^*) = \lambda \sum_{k \in \text{Case2}} \gamma^k (\tilde{Z}_k(p^*))^2. \tag{16}$$

The Weak Walras' Law says

$$0 \geq \sum_{k=1}^{N} p_k^* \tilde{Z}_k(p^*) = \sum_{k \in \text{Case1}} p_k^* \tilde{Z}_k(p^*) + \sum_{k \in \text{Case2}} p_k^* \tilde{Z}_k(p^*). \tag{17}$$

But for $k \in$ Case 1, $p_k^* \tilde{Z}_k(p^*) = 0$, and so

$$0 = \sum_{k \in \text{Case1}} p_k^* \tilde{Z}_k(p^*). \tag{18}$$

Therefore,

$$0 \geq \sum_{k \in \text{Case2}} p_k^* \tilde{Z}_k(p^*). \tag{19}$$

Hence, from (14.9) we have

$$0 \geq (1-\lambda) \cdot \sum_{k \in \text{Case2}} p_k^* \tilde{Z}_k(p^*) = \lambda \cdot \sum_{k \in \text{Case2}} \gamma^k (\tilde{Z}_k(p^*))^2. \tag{20}$$

The left-hand side $\leq 0$. But the right-hand side is necessarily nonnegative. It can be zero only if $\tilde{Z}_k(p^*) = 0$ for all $k$ such that $p_k^* > 0$ ($k$ in Case 2). Thus, $p^*$ is an equilibrium. This concludes the proof. **QED**