Lecture Notes, March 12 & 14

Sen's REVISED proof of the Group Contraction Lemma (this is the version in his notes "Arrow and the Impossibility Theorem")

Definition (Decisive Set): Let $x,y \in X$, $G \subseteq H$. G is decisive on (x, y) denoted $D_G(x,y)$ if [$x P_i y$ for all $i \in G$] implies [x P y] independent of $P_{j,}$, $j \in H$, $j \notin G$.

Note that under the Pareto Principle, there is always at least one decisive set, H. Further, the Field Expansion Lemma, says that a decisive set on one pair of possibilities is decisive on all.

Group Contraction Lemma: Let $G \subseteq H$, #G > 1, G decisive. Then there are G_1, G_2 , disjoint, nonempty, so that $G_1 \cup G_2 = G$, so that one of G_1, G_2 is decisive.

Proof: By Unrestricted Domain, we get to choose our example. Let

G_1 : x > y, x > z, y ? z
G_2 : x > y, z > y, x ? z
H \ G : unspecified

G is decisive so $D_G(x, y)$ so x P y .
Case 1: z P x

By transitivity of P, z P y, but then G_2 is decisive (only G_2's preferences on z/y were specified; G_2 is getting his way, independent of others' preferences). By the Field Expansion Lemma G_2 is decisive on everything.

Case 2: x P z

G_1 is getting his way. Only G_1's preferences on x/z were specified. So G_1 is decisive.

QED