Lecture Notes for February 8, 10, 13: Households

12.1 The structure of household consumption sets and preferences

Households are elements of the finite set H numbered $1, 2, \ldots, \#H$. A household $i \in H$ will be characterized by its possible consumption set $X^i \subseteq \mathbf{R}^N_+$, its preferences $\succeq_i$, and its endowment $r^i \in \mathbf{R}^N_+$. We will soon move to using a utility function $u^i(\cdot)$ to represent $\succeq_i$.

12.2 Consumption sets

(C.I) X^i is closed and nonempty.

(C.II) $X^i \subseteq \mathbf{R}^N_+$. X^i is unbounded above, that is, for any $x \in X^i$ there is $y \in X^i$ so that $y > x$, that is, for $n = 1, 2, ..., N, y_n \geq x_n$ and $y \neq x$.

(C.III) X^i is convex.

$X = \sum_{i \in H} X^i$.

12.2.1 Preferences

Each household $i \in H$ has a preference quasi-ordering on X^i, denoted $\succeq_i$. For typical $x, y \in X^i$, "$x \succeq_i y$" is read "x is preferred or indifferent to y (according to i)." We introduce the following terminology:

If $x \succeq_i y$ and $y \succeq_i x$ then $x \sim_i y$ ("x is indifferent to y"), If $x \succeq_i y$ but not $y \succeq_i x$ then $x \succ_i y$ ("x is strictly preferred to y").

We will assume $\succeq_i$ to be complete on X^i, that is, any two elements of X^i are comparable under $\succeq_i$. For all $x, y \in X^i$, $x \succeq_i y$, or $y \succeq_i x$ (or both). Since we take $\succeq_i$ to be a quasi-ordering, $\succeq_i$ is assumed to be transitive and reflexive.

The conventional alternative to describing the quasi-ordering $\succeq_i$ is to assume the presence of a utility function $u^i(x)$ so that $x \succeq_i y$ if and only if $u^i(x) \geq u^i(y)$. We will show below that the utility function can be derived from the quasi-ordering. Readers who prefer the utility function formulation may use it at will. Just read $u^i(x) \geq u^i(y)$ wherever you see $x \succeq_i y$.

12.2.2 Non-Satiation

(C.IV) (Non-Satiation) Let $x \in X^i$. Then there is $y \in X^i$ so that $y \succ_i x$.

12.2.3 Continuity

We now introduce the principal technical assumption on preferences, the assumption of continuity.

(C.V) (Continuity) For every $x^\circ \in X^i$, the sets
$A^i(x^\circ) = \{x \mid x \in X^i, x \succeq_i x^\circ\}$ and
$G^i(x^\circ) = \{x \mid x \in X^i, x^\circ \succeq_i x\}$ are closed.

Note that this assumption represents precisely what we would expect from a continuous utility function: that the inverse images of the closed sets $[0, a]$ and $[a, +\infty)$ are closed, where $a = u^i(x^\circ)$.

The following example represents an otherwise well-behaved preference ordering that is not continuous.

Example 12.1 (Lexicographic preferences) The lexicographic (dictionary-like) ordering on $\mathbf{R}^N$ (let's denote it $\succeq_L$) is described in the following way. Let

$x = (x_1, x_2, \ldots, x_N)$ and $y = (y_1, y_2, \ldots, y_N)$.

$$
\begin{aligned}
x \succ_L y & \text{if } x_1 > y_1, \text{ or} \\
& \text{if } x_1 = y_1 \text{ and } x_2 > y_2, \text{ or} \\
& \text{if } x_1 = y_1,\ x_2 = y_2, \text{ and } x_3 > y_3, \text{ and so forth} \ldots. \\
x \sim_L y & \text{if } x = y.
\end{aligned}
$$

$\succeq_L$ fulfills non-satiation, trivially fulfills strict convexity (C.VI(SC), introduced below), but does not fulfill continuity (C.V).

12.2.4 Attainable Consumption

Definition x is an ***attainable*** consumption if $y + r \geq x \geq 0$, where $y \in \mathcal{Y}$ and $r \in \mathbf{R}^N_+$ is the economy's initial resource endowment, so that y is an attainable production plan.

Note that the set of attainable consumptions is bounded under P.VI.

12.2.5 Convexity of preferences

(C.VI)(C) (Convexity of Preferences) $x \succ_i y$ implies $((1-\alpha)x + \alpha y) \succ_i y$, for $0 < \alpha < 1$.

(C.VI)(SC) (Strict Convexity of Preferences): Let $x \succeq_i y$, (note that this includes $x \sim_i y$), $x \neq y$, and let $0 < \alpha < 1$. Then $\alpha x + (1-\alpha)y \succ_i y$.

Equivalently, if preferences are characterized by a utility function $u^i(\cdot)$, then we can state C.VI(SC) as

$$u^i(x) \geq u^i(y), x \neq y, \text{ implies } u^i[\alpha x + (1-\alpha)y] > u^i(y).$$

An immediate consequence of C.VI(C) is that $A^i(x^\circ)$ is convex for every $x^\circ \in X^i$.

12.3 Representation of $\succeq_i$: Existence of a continuous utility function

Definition Let u^i: $X^i \to \mathbf{R}$. $u^i(\cdot)$ is a utility function that ***represents*** the preference ordering $\succeq_i$ if for all $x, y \in X^i$, $u^i(x) \geq u^i(y)$ if and only if $x \succeq_i y$. This implies that $u^i(x) > u^i(y)$ if and only if $x \succ_i y$.

12.3.1 Weak Conditions for Existence of a Continuous Utility Function

Theorem 12.1 Let $\succeq_i, X^i$, fulfill C.I, C.II, C.III, C.V. Then there is $u^i : X^i \to \mathrm{R}, u^i(\cdot)$ continuous on X^i, so that $u^i(\cdot)$ is a utility function representing $\succeq_i$.

Proof See Debreu (1959, Section 4.6) or Debreu (1954). QED

12.3.2 Construction of a continuous utility function

Shortcut: use weak desirability, $X^i = R_+^N$ and a 45° line.

12.4 Choice and boundedness of budget sets, $\tilde{\boldsymbol{B}}^{\boldsymbol{i}}(\boldsymbol{p})$

Choose $c \in \mathbf{R}_+$ so that $|x| < c$ (a strict inequality) for all attainable consumptions x. Choose c sufficiently large that $X^i \cap \{x \mid x \in \mathbf{R}^N, c > |x|\} \neq \phi$;

$$\tilde{B}^i(p) = \{x \mid x \in \mathbf{R}^N, p \cdot x \leq \tilde{M}^i(p)\} \cap \{x||x| \leq c\}.$$

$$\begin{aligned}\tilde{D}^i(p) &\equiv \{x \mid x \in \tilde{B}^i(p) \cap X^i, x \succeq_i y \text{ for all } y \in \tilde{B}^i(p) \cap X^i\} \\ &\equiv \{x \mid x \in \tilde{B}^i(p) \cap X^i, x \text{ maximizes } u^i(y) \text{ for all } y \in \tilde{B}^i(p) \cap X^i\}.\end{aligned}$$

To characterize market demand let

$$\tilde{D}(p) = \sum_{i \in H} \tilde{D}^i(p).$$

Lemma 12.1 $\tilde{B}^i(p)$ is a closed set.

We will restrict attention to models where $\tilde{M}^i(p)$ is homogeneous of degree one, that is, where $\tilde{M}^i(\lambda p) = \lambda \tilde{M}^i(p)$. It is immediate then that $\tilde{B}^i(p)$ is homogeneous of degree zero.

Lemma 12.2 Let $\tilde{M}^i(p)$ be homogeneous of degree 1. Let $\tilde{B}^i(p)$ and $\tilde{D}^i(p)$ $\neq \emptyset$. Then $\tilde{B}^i(p)$ and $\tilde{D}^i(p)$ are homogeneous of degree 0.

$$P \equiv \left\{ p \mid p \in \mathbf{R}^N, p_n \geq 0, n = 1, 2, 3, \ldots, N, \ \sum_{n=1}^{N} p_n = 1 \right\}.$$

12.4.1 Adequacy of income

Continuity of demand behavior may require sufficient income for the household to keep the budget set from coinciding with the boundary of X^i.

(C.VII) For all $i \in H$, $\tilde{M}^i(p) > \inf_{x \in X^i \cap \{x \mid |x| \leq c\}} p \cdot x$ for all $p \in P$.

The example below demonstrates that when (C.VII) is not fulfilled, demand behavior may be discontinuous.

Example 12.2 [The Arrow Corner]

$$X^i = \mathbf{R}^2_+,$$
$$r^i = (1, 0),$$
$$\tilde{M}^i(p) = p \cdot r^i.$$

Let $p^\circ = (0, 1)$. Then

$$\tilde{B}^i(p^\circ) \cap X^i = \{(x, y) \mid c \geq x \geq 0, y = 0\},$$

the truncated nonnegative x axis. Consider the sequence $p^\nu = (1/\nu, 1 - 1/\nu)$.

$p^\nu \to p^\circ$. We have

$$\tilde{B}^i(p^\nu) \cap X^i = \left\{(x,y) \mid p^\nu \cdot (x,y) \le \frac{1}{\nu}, (x,y) \ge 0, c \ge |(x,y)| \ge 0\right\},$$

$(c,0) \in \tilde{B}^i(p^\circ)$, but there is no sequence $(x^\nu, y^\nu) \in \tilde{B}^i(p^\nu)$ so that $(x^\nu, y^\nu) \to (c,0)$. On the contrary, for any sequence $(x^\nu, y^\nu) \in \tilde{B}^i(p^\nu)$ so that $(x^\nu, y^\nu) = \tilde{D}^i(p^\nu)$, (x^ν, y^ν) will converge to some $(x^*, 0)$, where $0 \le x^* \le 1$. For suitably chosen $\succeq_i$, we may have $(c,0) = \tilde{D}^i(p^\circ)$. Hence $\tilde{D}^i(p)$ need not be continuous at p°. This completes the example.

12.5 Demand behavior under strict convexity

Theorem 12.2 Assume C.I–C.V, C.VI(SC), and C.VII. Let $\tilde{M}^i(p)$ be a continuous function for all $p \in P$. Then $\tilde{D}^i(p)$ is a well-defined, point-valued, continuous function for all $p \in P$.

Proving well-defined, point-valued, is easy. Proving continuity is hard, because the proof necessarily involves C.VII. Continuity is proved by contradiction; we know the proof is going to be tricky.

Proof $\tilde{B}^i(p) \cap X^i$ is the intersection of the closed set $\{x \mid p \cdot x \le \tilde{M}^i(p)\}$ with the compact set $\{x \mid |x| \le c\}$ and the closed set X^i. Hence it is compact. It is nonempty by C.VII. Because $\tilde{D}^i(p)$ is characterized by the maximization of a continuous function, $u^i(\cdot)$, on this compact nonempty set, there is a well-defined maximum value, $u^* = u^i(x^*)$, where x^* is the utility-optimizing value of x in $\tilde{B}^i(p) \cap X^i$. We must show that x^* is unique for each $p \in P$ and that x^* is a continuous function of p.

We will now demonstrate that uniqueness follows from strict convexity of preferences (C.VI(SC)). Suppose there is $x' \in \tilde{B}^i(p) \cap X^i, x' \ne x^*, x' \sim_i x^*$. We must show that this leads to a contradiction. But now consider a convex combination of x' and x^*. Choose $0 < \alpha < 1$. The point $\alpha x' + (1-\alpha)x^* \in \tilde{B}^i(p) \cap X^i$ by convexity of X^i and $\tilde{B}^i(p)$. But C.VI(SC), strict convexity of preferences, implies that $[\alpha x' + (1-\alpha)x^*] \succ_i x' \sim_i x^*$. This is a contradiction, since x^* and x' are elements of $\tilde{D}^i(p)$. Hence x^* is the unique element of $\tilde{D}^i(p)$. We can now, without loss of generality, refer to $\tilde{D}^i(p)$ as a (point-valued) function.

To demonstrate continuity, let $p^\nu \in P$, $\nu = 1, 2, 3, \ldots, p^\nu \to p^\circ$. We must show that $\tilde{D}^i(p^\nu) \to \tilde{D}^i(p^\circ)$. $\tilde{D}^i(p^\nu)$ is a sequence in a compact set. Without loss of generality take a convergent subsequence, $\tilde{D}^i(p^\nu) \to x^\circ$. We must show that $x^\circ = \tilde{D}^i(p^\circ)$. We will use a proof by contradiction.

Define

$$\hat{x} = \underset{x \in X^i \cap \{y | y \in \mathbf{R}^N, c \geq |y|\}}{\arg\min} p^\circ \cdot x.$$

The expression "$\hat{x} = \arg\min_{x \in X^i \cap \{y | y \in \mathbf{R}^N, c \geq |y|\}} p^\circ \cdot x$" defines $\hat{x}$ as the minimizer of $p^\circ \cdot x$ in the domain $X^i \cap \{y \mid y \in \mathbf{R}^N, c \geq |y|\}$. $\hat{x}$ is well defined (though it may not be unique) since it represents a minimum of a continuous function taken over a compact domain.

Now consider two cases. In each case we will construct a sequence w^ν in $X^i \cap \{y \mid y \in \mathbf{R}^N, c \geq |y|\}$.

Case 1: If $p^\circ \cdot \tilde{D}^i(p^\circ) < \tilde{M}^i(p^\circ)$ for ν large $p^\nu \cdot \tilde{D}^i(p^\circ) < \tilde{M}^i(p^\nu)$. Then let $w^\nu = D^i(p^\circ)$.

Case 2: If $p^\circ \cdot \tilde{D}^i(p^\circ) = \tilde{M}^i(p^\circ)$ then by (C.VII) $p^\circ \cdot \tilde{D}^i(p^\circ) > p^\circ \cdot \hat{x}$.

Let

$$\alpha^\nu = \min\left[1, \frac{\tilde{M}^i(p^\nu) - p^\nu \cdot \hat{x}}{p^\nu \cdot (\tilde{D}^i(p^\circ) - \hat{x})}\right].$$

For ν large, the denominator is positive, α^ν is well defined (this is where C.VII enters the proof), and $0 \leq \alpha^\nu \leq 1$. Let $w^\nu = (1-\alpha^\nu)\hat{x} + \alpha^\nu \tilde{D}^i(p^\circ)$.

Note that $\tilde{M}^i(p)$ is continuous in p. The fraction in the definition of α^ν is the proportion of the move from $\hat{x}$ to $\tilde{D}^i(p^\circ)$ that the household can afford at prices p^ν. As ν becomes large, the proportion approaches or exceeds unity.

Then in both Case 1 and Case 2, $w^\nu \to \tilde{D}^i(p^\circ)$ and $w^\nu \in \tilde{B}^i(p^\nu) \cap X^i$. Suppose, contrary to the theorem, $x^\circ \neq \tilde{D}^i(p^\circ)$. Then $u^i(x^\circ) < u^i(\tilde{D}^i(p^\circ))$. But u^i is continuous, so $u^i(\tilde{D}^i(p^\nu) \to u^i(x^\circ)$ and $u^i(w^\nu) \to u^i(\tilde{D}^i(p^\circ))$. Thus, for ν large, $u^i(w^\nu) > u^i(\tilde{D}^i(p^\nu))$. But this is a contradiction, since $\tilde{D}^i(p^\nu)$ maximizes $u^i(\cdot)$ in $\tilde{B}^i(p^\nu) \cap X^i$. The contradiction proves the result. This completes the demonstration of continuity. QED

Theorem 12.2 gives a family of sufficient conditions for demand behavior of the household to be very well behaved. It will be a continuous (point-valued) function of prices if preferences are continuous and strictly convex and if income is a continuous function of prices and sufficiently positive.

What will household spending patterns look like? What is the value of household expenditures, $p \cdot \tilde{D}^i(p)$? There are two significant constraints on $p \cdot \tilde{D}^i(p)$, budget and length: $p \cdot \tilde{D}^i(p) \leq \tilde{M}^i(p)$ and $|\tilde{D}^i(p)| \leq c$. In addition, of course, $\tilde{D}^i(p)$ must optimize consumption choice with regard to preferences $\succeq_i$ or equivalently with regard to the utility function $u^i(\cdot)$. We have enough structure on preferences and the budget set to actually say a fair amount about the character of spending and where $\tilde{D}^i(p)$ is located. This is embodied in

Lemma 12.3 Assume C.I–C.V, C.VI(C), and C.VII. Then $p \cdot \tilde{D}^i(p) \leq \tilde{M}^i(p)$. Further, if $p \cdot \tilde{D}^i(p) < \tilde{M}^i(p)$ then $|\tilde{D}^i(p)| = c$.

Proof $\tilde{D}^i(p) \in \tilde{B}^i(p)$ by definition. However, that ensures $p \cdot \tilde{D}^i(p) \leq \tilde{M}^i(p)$ and hence the weak inequality surely holds. Suppose, however, $p \cdot \tilde{D}^i(p) <$

$\tilde{M}^i(p)$ and $|\tilde{D}^i(p)| < c$. We wish to show that this leads to a contradiction. Recall C.IV (Non-Satiation) and C.VI(C) (Convexity). By C.IV there is $w^* \in X^i$ so that $w^* \succ_i \tilde{D}^i(p)$. Clearly, $w^* \notin \tilde{B}^i(p)$ so one (or both) of two conditions holds: (a) $p \cdot w^* > \tilde{M}^i(p)$, (b) $|w^*| > c$.

Set $w' = \alpha w^* + (1 - \alpha)\tilde{D}^i(p)$. There is an $\alpha(1 > \alpha > 0)$ sufficiently small so that $p \cdot w' \leq \tilde{M}^i(p)$ and $|w'| \leq c$. Thus $w' \in \tilde{B}^i(p)$. Now $w' \succ_i \tilde{D}^i(p)$ by C.VI(C), which is a contradiction since $\tilde{D}^i(p)$ is the preference optimizer in $\tilde{B}^i(p)$. The contradiction shows that we cannot have both $p \cdot \tilde{D}^i(p) < \tilde{M}^i(p)$ and $|\tilde{D}^i(p)| < c$. Hence, if the first inequality holds, we must have $|\tilde{D}^i(p)| = c$. QED