13 Social choice

So far, all of our models involved a single decision maker. An important, perhaps *the* important, question for economics is whether the desires and wants of various agents can be rationally aggregated to a social ranking over outcomes.

Fix a set X of alternatives and a set I of agents.

$$\mathcal{B} = 2^{X \times X}$$

is the collection of all binary relations on X.

$$\mathcal{R} = \{\succsim \subseteq X \times X : \succsim \text{ is complete and transitive}\}$$

is the collection of all preference relations on X.

$$\mathcal{R}^I = \{(\succsim_1, \dots, \succsim_I) : \succsim_i \in \mathcal{R}\} = \prod_{i \in I} \mathcal{R}$$

is the collection of all preference profiles, i.e. $\succsim_i$ denotes agent i's preferences over X. We will let $\boldsymbol{\succsim} = (\succsim_1, \succsim_2, \dots, \succsim_I)$ denote a generic profile of preference relations.

$\mathcal{P} = \{\succsim \in \mathcal{R} : \text{not } x \sim y, \text{ for all } x, y \in X\}$ is the collection of all strict preference relations on X. $\mathcal{P}^I$ is the set of all strict preference profiles.

Definition 13.1. A *social welfare functional* is a function $F : \mathcal{A} \to \mathcal{B}$, where $\mathcal{A} \subseteq \mathcal{R}^I$.

A social welfare functional aggregates the preferences of the I individuals. It takes each preference profile and assigns to it a binary relation over X. We take $F(\boldsymbol{\succsim})$ to mean the social planner's or the government's preferences over X when individuals' preferences are $\boldsymbol{\succsim}$. Note that society's preferences are not necessarily complete or transitive, since the target set $\mathcal{B}$ is the collection of all binary relations, including those that are not complete or not transitive.

One example of a social welfare functional is the Associated Press college football poll. Each week, every member of the poll submits a ranked list of all the college football teams in the National Collegiate Athletic Association. Then X is the set of all NCAA football teams, and I is the set of all writers who are polled. The Associated Press has an algorithm F which inputs a profile of ballots from every writer $\boldsymbol{\succsim}$ and outputs a single national ranking $F(\boldsymbol{\succsim})$. The poll changes every week, because writers submit new ballots every week, so the algorithm must be well-defined for arbitrary profiles of ballots.

Let F_p denote the strict component of F, i.e. $xF_p(\boldsymbol{\succsim})y$ if and only if $xF(\boldsymbol{\succsim})y$ and not $yF(\boldsymbol{\succsim})x$.

Definition 13.2. A social welfare function F is *rational* if $F(\boldsymbol{\succsim})$ is complete and transitive for all $\boldsymbol{\succsim} \in \mathcal{A}$, i.e. if $F(\boldsymbol{\succsim}) \in \mathcal{R}$ for all $\boldsymbol{\succsim} \in \mathcal{A}$.

Example 13.3. Define the social welfare function $F : \mathcal{R}^I \to \mathcal{B}$ by

$$xF(\boldsymbol{\succsim})y \Leftrightarrow |\{i \in I : x \succsim_i y\}| \geq |\{i \in I : y \succsim_i x\}|.$$

This corresponds to majority rule. Let $I = \{1, 2, 3\}$. Consider any preference profile $\succsim$ such that:

$$\begin{aligned} a \succ_1 b \succ_1 c \\ b \succ_2 a \succ_2 c \\ c \succ_3 a \succ_3 b \end{aligned} .$$

Then $aF_p(\succsim)b$, $bF_p(\succsim)c$, and $cF_p(\succsim)a$. Thus $F(\succsim)$ is not transitive.

Definition 13.4. A social welfare functional F is *weakly Paretian* if, $x \succ_i y$ for all $i \in I$, then $xF_p(\succsim_1, \ldots, \succsim_I)y$.

This is a weak notion of efficiency. If everyone strictly prefers outcome x to outcome y, then society should also strictly prefer x to y. Notice that if any individual's preference is weak, but not strict, the conclusion does not have to hold.

Definition 13.5. A social welfare functional F is *independent of irrelevant alternatives* if

$$x \succsim_i y \Leftrightarrow x \succsim_i' y \text{ and } y \succsim_i x \Leftrightarrow y \succsim_i' x$$

for all $i \in I$ implies

$$xF(\succsim)y \Leftrightarrow xF(\succsim')y \text{ and } yF(\succsim)x \Leftrightarrow yF(\succsim')x,$$

for all $x, y \in X$.

This condition is more controversial. Roughly speaking, social preference over a pair x and y should depend only on individuals' preferences over x and y.

Example 13.6. Define F by

$$xF(\succsim)y \Leftrightarrow \sum_{i \in I} |\{z : x \succsim_i z\}| \geq \sum_{i \in I} |\{z : y \succsim_i z\}|.$$

This is known as the Borda count. This is also essentially the social welfare functional used by the Associated Press to compute its national rankings each week. Suppose $X = \{a, b, c\}$ and $I = \{1, 2\}$ and consider the preference profile $\succsim$ defined by

$$\begin{aligned} a \succ_1 b \succ_1 c \\ c \succ_2 b \succ_2 a \end{aligned} .$$

Then $\sum_{i \in I} |\{z : a \succsim_i z\}| = 3 + 1 = 2 + 2 = \sum_{i \in I} |\{z : a \succsim_i z\}|$. So $bF(\succsim)a$. Now consider the preference profile $\succsim'$ defined by

$$\begin{aligned} a \succ_1' c \succ_1' b \\ c \succ_2' b \succ_2' a \end{aligned} .$$

(Perhaps team c won a big game that week and voter 1 decided to move c up on her ballot.) Then $\sum_{i \in I} |\{z : a \succsim_i z\} = 3 + 1 > 1 + 2 = \sum_{i \in I} |\{z : a \succsim_i z\}$. So $aF_p(\succsim')b$. But both $\succsim$ and $\succsim'$ agree on a and b, so this violates independence of irrelevant alternatives.

Definition 13.7. A social welfare functional F is *dictatorial* if there exists $d \in I$ such that $xF_p(\succsim)y$ for all $\succsim \in \mathcal{A}$ such that $x \succ_d y$. It is *nondictatorial* if it is not dictatorial.

The previous conditions seem very desirable for any rule to aggregate preferences. Unfortunately, the following result asserts that there is no social welfare functional which is well-defined for all possible preference profiles which is rational, weakly Paretian, independent of irrelevant alternatives, and nondictatorial.

Theorem 13.8 (General Possibility Theorem, Arrow)**.** *Suppose $|X| \geq 3$ and $\mathcal{A} = \mathcal{R}^I$ or $\mathcal{A} = \mathcal{P}^I$.*[33] *If the social choice functional $F : \mathcal{A} \to \mathcal{R}$ is rational, weakly Paretian, and independent of irrelevant alternatives, then F is dictatorial.*

Definition 13.9. Given a set I, a nonempty collection of subsets $\mathcal{U} \subseteq 2^I$ is an *ultrafilter* of I if:

1. $\emptyset \notin \mathcal{U}$;
2. If $A, B \in \mathcal{U}$, then $A \cap B \in \mathcal{U}$;
3. If $A \in \mathcal{U}$ and $A \subseteq B \subseteq I$, then $B \in \mathcal{U}$;
4. If $A \notin \mathcal{U}$, then $I \setminus A \in \mathcal{U}$.

In fact, condition (3) is redundant; it follows from properties (1), (2), and (4).[34] To see this, let $A \in \mathcal{U}$ and $A \subseteq B$. By way of contradiction, suppose $B \notin \mathcal{U}$, then $I \setminus A \in \mathcal{U}$, by (4). Then, applying (2), $A \cap (I \setminus A) = \emptyset \in \mathcal{U}$, which contradicts property (1). Thus B must be decisive.

Lemma 13.10. *Suppose I is finite. If $\mathcal{U}$ is an ultrafilter of I, then there exists some $i \in I$ such that $\{i\} \in \mathcal{U}$.*

Proof. Let $A^* = \bigcap_{A \in \mathcal{U}} A$. Since $\mathcal{U}$ is finite, repeated applications of property (2) imply that $A^* \in \mathcal{U}$. Property (1) implies A^* is nonempty. So there exists some $i \in A^*$. We prove that $\{i\} \in \mathcal{U}$ by contradiction. Suppose $\{i\} \notin \mathcal{U}$. Then $I \setminus \{i\} \in \mathcal{U}$ by property (4). This contradicts $i \in \bigcap_{A \in \mathcal{U}} A$, because $i \notin I \setminus \{i\}$. Hence $\{i\} \in \mathcal{U}$. $\square$

Definition 13.11. Given a social welfare functional F, a subset $S \subseteq I$ is:

- *decisive for x over y* if

$$\left.\begin{array}{ll} x \succ_i y & \text{for all } i \in S \\ y \succ_j x & \text{for all } j \notin S \end{array}\right\} \Longrightarrow xF_p(\succsim)y;$$

- *decisive* if it is decisive for x over y for all $x, y \in X$.

[33] The assumption that $\mathcal{A} = \mathcal{R}^I$ is sometimes called the full domain or unrestricted domain assumption.
[34] This redundant condition is usually included because ultrafilters are particular filters, which satisfy the first three conditions.

So, a coalition S is decisive for x over y if $xF_p(\succsim)y$ whenever $x \succ_i y$ for all $i \in S$ and $y \succ_j x$ for all $j \notin S$.

Proof of Theorem 13.8. We will prove the result for the case where $\mathcal{A} = \mathcal{P}^I$.

We first prove that if S is decisive for x over y, then S is decisive. Next, we will show that the collection of decisive subsets is an ultrafilter. Then Lemma 13.10 implies there exists a singleton which is decisive. That singleton will be the desired dictator.

Step 1: Suppose S is decisive for x over y. If $z \neq x$ or $z \neq y$, then S is decisive for x over z and for z over y. First, suppose $z \neq x$ or $z \neq y$. If either $z = x$ or $z = y$, the desired conclusions follow immediately. So, assume without loss of generality that x, y, z are distinct. We will go through this step carefully, because the same technique is used throughout the proof. Fix any preference profile $\succsim$ where

$$\begin{array}{ll} x \succ_i y \succ_i z & \text{for all } i \in S \\ y \succ_j z \succ_j x & \text{for all } j \notin S \end{array}.$$

Such a preference profile exists because $\mathcal{A} = \mathcal{R}^I$ or $\mathcal{A} = \mathcal{P}^I$. Since S is decisive for x over y,

$$\begin{array}{ll} x \succ_i y & \text{for all } i \in S \\ y \succ_j x & \text{for all } j \notin S \end{array} \Rightarrow xF_p(\succsim)y.$$

By the weak Pareto axiom,

$$y \succ_k z \text{ for all } k \in I \Rightarrow yF_p(\succsim)z.$$

By transitivity,

$$xF_p(\succsim)y \text{ and } yF_p(\succsim)z \Rightarrow xF_p(\succsim)z.$$

By independence of irrelevant alternatives, for any $\succsim' = (\succsim'_1, \ldots, \succsim'_I)$,

$$\begin{array}{ll} x \succ'_i z & \text{for all } i \in S \\ z \succ'_j x & \text{for all } j \notin S \end{array} \Rightarrow xF_p(\succsim')z,$$

because the ranking of x and z is the same in both $\succsim$ and $\succsim'$ for all individuals. This proves S is decisive for x over z.

The proof that S is also decisive for z over y is similar, using any preference profile $\succsim$ such that

$$\begin{array}{ll} z \succ_i x \succ_i y & \text{for all } i \in S \\ y \succ_j z \succ_j x & \text{for all } j \in I \setminus S \end{array}.$$

Step 2: If S is decisive for x over y, then S is decisive. Let $z, z' \in X$. Without loss of generality, suppose $z \neq x$ and $z' \neq y$, since otherwise we would be done by Step 1. Then, by Step 1, S is decisive for z over y. Then, applying Step 1 to the fact S is decisive for z over y, we have S is decisive for z over z'.

Step 3: Let $\mathcal{U} = \{S \in I : S$ is decisive$\}$. $\mathcal{U}$ is an ultrafilter of I. Since F is weakly Paretian, if $x \succ_i y$ for all $i \in I$, then $xF_p(\succsim)y$. So I is decisive, hence $\mathcal{U}$ is nonempty. Similarly, if F is weakly Paretian, then the empty set cannot be decisive, so $\emptyset \notin \mathcal{U}$. Hence property (1) is met.

We now check property (2), the intersection of two decisive coalitions is decisive. Suppose S and T are decisive. Then consider any preference profile $\succsim$ such that

$$\begin{array}{ll} x \succ_i z \succ_i y & \text{if } i \in S \text{ and } i \in T \\ z \succ_i y \succ_i x & \text{if } i \in S \text{ and } i \notin T \\ y \succ_i x \succ_i z & \text{if } i \notin S \text{ and } i \in T \\ y \succ_i z \succ_i x & \text{if } i \notin S \text{ and } i \notin T \end{array}.$$

T is decisive, so $x \succ_i z$ for all $i \in T$ and $z \succ_j x$ for all $j \notin T$ imply $xF_p(\succsim)z$. Similarly, S is decisive, so $z \succ_i y$ for all $i \in S$ and $y \succ_j z$ for all $j \notin S$ imply $zF_p(\succsim)y$. By transitivity, $xF_p(\succsim)y$. By independence of irrelevant alternatives, $xF_p(\succsim')y$ whenever $x \succ'_i y$ for all $i \in S \cap T$ and $y \succ'_j x$ for all $j \notin S \cap T$. By Step 2, this suffices to show $S \cap T$ is decisive. This proves property (2) of the definition.

Recall that condition (3) is redundant, so it is now sufficient to check property (4), that for any coalition, either it or its complement must be decisive. Suppose S is not decisive. Then there exists x, y and some preference profile $\succsim$ such that

$$\begin{array}{ll} x \succ_i y & \text{for all } i \in S \\ y \succ_j x & \text{for all } j \in I \setminus S \end{array},$$

and $yF(\succsim)x$. Now fix a preference profile $\succsim'$ such that

$$\begin{array}{ll} x \succ'_i z \succ'_i y & \text{for all } i \in S \\ y \succ'_j x \succ'_j z & \text{for all } j \in I \setminus S \end{array}.$$

By independence of irrelevant alternatives, $yF(\succsim')x$. By the weak Pareto condition, $xF_p(\succsim')z$. Transitivity implies $yF_p(\succsim')z$. By independence of irrelevant alternatives, $yF_p(\succsim^*)z$ for any preference profile $\succsim^*$ such that $y \succ^*_j z$ for all $j \in I \setminus S$ and $z \succ^*_i y$ for all $i \in S$. Thus $I \setminus S$ is decisive for y over z, hence decisive, by Step 2. This proves property (4).

Finally, we check property (3), that the superset of any decisive coalition is decisive.

Step 4: There exists a dictator. By Step 3 and Lemma 13.10, there exists a $\{d\}$ such that d is decisive. Pick any $x, y \in X$. Fix some preference profile $\succsim$ such that $x \succ_d y$. Let $S = \{i : x \succ_i y\}$. For $j \in I \setminus S$, consider any preference profile $\succsim'$ such that

$$\begin{array}{ll} x \succ'_i z \succ'_i y & \text{for all } i \in S \\ z \succ'_j y \succ'_j x & \text{for all } j \in I \setminus S \end{array}.$$

Since $\{d\} \subseteq S$ and $\{d\}$ is decisive, S is decisive for x over z, so $xF_p(\succsim')z$. By the weak Pareto axiom, $zF_p(\succsim')y$. By transitivity, $xF_p(\succsim')y$. By independence of irrelevant alternatives, $xF_p(\succsim)y$, because $\succsim$ and $\succsim'$ agree on the ranking of x and y for each individual. Since x, y are arbitrary, this proves d is a dictator. □

The following is optional.

Definition 13.12. A *social choice function* is a function $f : \mathcal{A} \to X$, where $\mathcal{A} \subseteq \mathcal{R}^I$.

Definition 13.13. A social choice function f is *weakly Paretian* if $x \succ_i y$ for all $i \in I$ implies $y \neq f(\succsim_1, \ldots, \succsim_I)$.

Definition 13.14. A social choice function f is *monotonic* if $\{y \in X : f(\succsim_1, \ldots, \succsim_I) \succsim_i y\} \subseteq \{y : f(\succsim_1, \ldots, \succsim_I) \succsim_i' y\}$ for all $i \in I$ implies $f(\succsim_1, \ldots, \succsim_I) = f(\succsim_1', \ldots, \succsim_I')$.

Definition 13.15. A social choice function f is *dictatorial* if there exists $d \in I$ such that $f(\succsim_1, \ldots, \succsim_I) \in C_{\succsim_d}(X)$ for all $(\succsim_1, \ldots, \succsim_I) \in \mathcal{A}$.

Theorem 13.16. *Suppose $|X| \geq 3$ and $\mathcal{A} = \mathcal{R}^I$ or $\mathcal{A} = \mathcal{P}^I$. If the social choice function $f : \mathcal{A} \to X$ is weakly Paretian and monotonic, then f is dictatorial.*

Definition 13.17. A social choice function $f : \mathcal{P}^I \to X$ is *incentive compatible* if

$$f(\succsim_1, \ldots, \succsim_{h-1}, \succsim_h, \succsim_{h+1}, \ldots, \succsim_I) \succsim_h f(\succsim_1, \ldots, \succsim_{h-1}, \succsim_h', \succsim_{h+1}, \ldots, \succsim_I)$$

for all $h \in I$, $(\succsim_1, \ldots, \succsim_I) \in \mathcal{P}^I$, and $\succsim_h' \in \mathcal{P}$.

Theorem 13.18. *Suppose $|X| \geq 3$. If the social choice function $f : \mathcal{P}^I \to X$ is weakly Paretian and incentive compatible, then f is dictatorial.*