Lecture Notes, February 23, 25, 2010

Social Choice Theory, Arrow Possibility Theorem

Bergson-Samuelson social welfare function $W(u^1(x^1), u^2(x^2), \ldots, u^{\#H}(x^{\#H}))$ with $\frac{\partial W}{\partial u^i} > 0$ all i.

Let the allocation $x^* \in R^{N(\#H)}_+$ maximize W subject to the usual technology constraints. Then x* is a Pareto efficient allocation.

Further, suppose $x^{**} \in R^{N(\#H)}_+$ is a Pareto efficient allocation. Then there is a specification of W so that x** maximizes W subject to constraint.

Paradox of Voting (Condorcet)

Cyclic majority:

Voter preferences:

1	2	3
A	B	C
B	C	A
C	A	B

Majority votes A > B, B > C. Transitivity requires A > C but majority votes C > A.

Conclusion: Majority voting on pairwise alternatives by rational (transitive) agents can give rise to intransitive group preferences.

Is this an anomaly? Or systemic. Arrow Possibility Theorem says systemic.

Arrow (Im) Possibility Theorem:

We'll follow Sen's treatment. For simplicity we'll deal in strong orderings (strict preference) only

X = Space of alternative choices

Π = Space of transitive strict orderings on X

H = Set of voters, numbered #H

$\Pi^{\#H}$ = #H - fold Cartesian product of Π , space of preference profiles

f: $\Pi^{\#H} \rightarrow \Pi$, f is an Arrow Social Welfare Function.

P_i represents the preference ordering of typical household i. $\{P_i\}$ represents a preference profile, $\{P_i\} \in \Pi^{\#H}$. P represents the resulting group (social) ordering.
" x P_i y " is read "x is preferred to y by i" for $i \in H$

P (without subscript) denotes the social ordering, $f(P_1, P_2, ..., P_{\#H})$.

Unrestricted Domain: Π = all logically possible strict orderings on X.
$\Pi^{\#H}$ = all logically possible combinations of #H elements of Π.

Non-Dictatorship: There is no $j \in H$, so that $x P y \Leftrightarrow x P_j y$, for all $x, y \in X$, for all $\{P_i\} \in \Pi^{\#H}$.

(Weak) Pareto Principle: Let x P_i y for all $i \in H$. Then x P y.

For $S \subseteq X$, Define $C(S) = \{ x \mid x \in S, \ x P y, \text{ for all } y \in S, y \neq x \}$

Independence of Irrelevant Alternatives: Let $\{P_i\} \in \Pi^{\#H}$ and $\{P'_i\} \in \Pi^{\#H}$, so that for all $x, y \in S \subseteq X$, x P_i y if and only if ($\Leftrightarrow$) x P'_i y. Then $C(S) = C'(S)$.

General Possibility Theorem (Arrow): Let f satisfy (Weak) Pareto Principle, Independence of Irrelevant Alternatives, Unrestricted Domain, and let #H be finite, $\#X \geq 3$. Then there is a dictator; there is no f satisfying non-dictatorship and the three other conditions.

Definition (Decisive Set): Let x,y $\in$ X, G $\subseteq$ H. G is decisive on (x, y) denoted $\overline{D}_G(x,y)$ if [x P_i y for all i $\in$ G] implies [x P y] independent of P_j, , j $\in$ H, j $\notin$ G.

Definition (Almost Decisive Set): Let x,y $\in$ X, G $\subseteq$ H. G is almost decisive on (x, y) denoted D_G(x, y) if [x P_i y for all i $\in$ G; y P_j x for all j $\notin$ G] implies [x P y] .

Note: $\overline{D}_G(x,y)$ implies D(x, y) but D(x,y) does not imply $\overline{D}_G(x,y)$ (though it does not contradict either).

Field Expansion Lemma: Assume (Weak) Pareto Principle, Independence of Irrelevant Alternatives, Unrestricted Domain, Non-Dictatorship. Let x, y $\in$ X, G $\subseteq$ H , D_G(x, y). Then for arbitrary a, b $\in$ X, a $\neq$ b , $\overline{D}_G(a,b)$.

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Proof: Introduce a, b $\in$ X, a $\neq$ b. We'll consider three cases
1. x $\neq$ a $\neq$ y, x $\neq$ b $\neq$ y
2. a = x. This is typical of the three other cases (which we'll skip, assuming their treatments are symmetric) b = x, a = y, b = y.
3. a = x and b = y.

Case 1 (a, b, x, y are all distinct) : Let G have preferences : a > x > y > b .
Unrestricted Domain allows us to make this choice. Let H \ G have preferences: a > x, y > b,
y > x, a ? b (unspecified).
Pareto implies a P x, y P b.
D_G(x, y) implies x P y.
P transitive implies a P b, independent of H \ G's preferences.
Independence implies $\overline{D}_G(a,b)$.

Case 2 (a = x): Let G have preferences: a > y > b. Let H\G have preferences: y > a, y > b, a ? b (unspecified). D_G(x, y) implies that xPy or equivalently aPy. Pareto principle implies yPb. Transitivity implies aPb. By Independence, then $\overline{D}_G(a,b)$.

Case 3 (a = x, b = y): Introduce a third state z, distinct from a and b, x and y. Since #X ≥ 3, this is possible. We now consider a succession of examples.

Let G have preferences: (x=)a > (y=)b > z. Let H\G have preferences: b >a, b > z, a?z (unspecified). D_G(x, y) implies that xPy or equivalently aPb. Pareto principle implies bPz. Transitivity implies (x=)aPz. By Independence, then $\overline{D}_G$ (x, z).

Now consider G: b >x > z ; H\G: b ?z, z?x (unspecified), b > x. We have xPz by $\overline{D}_G$ (x,z). By Pareto we have bPx. By transitivity we have (y=)bPz. By Independence, then $\overline{D}_G$ (y, z). [Is this step necessary?]

Now consider G: y(=b) > z > x(=a); H\G z>x, x?y, z?y. $\overline{D}_G$ (y, z) implies yPz. Pareto implies zPx. Transitivity implies yPx. Independence implies $\overline{D}_G$ (y, x) = $\overline{D}_G$ (b, a). [Is this step necessary?]

Repeating the argument in Case 2, consider G: a(=x) > z > b(=y). Let H\G have preferences: z > a, z > b, a ? b (unspecified). $\overline{D}_G$ (x, z) implies xPz. Pareto implies zPb. Transitivity implies x(=a)Pb. Independence implies $\overline{D}_G$ (a, b) = $\overline{D}_G$ (x, y).

QED

The Field Expansion Lemma tells us that a set that is almost decisive on any (x, y), x ≠ y, is decisive on arbitrary (a, b).

Note that under the Pareto Principle, there is always at least one decisive set, H.

<u>Group Contraction Lemma:</u> Let G ⊆ H, #G > 1, G decisive. Then there are G_1, G_2, disjoint, nonempty, so that $G_1 \cup G_2 = G$, so that one of G_1, G_2 is decisive.

<u>Proof</u>: By Unrestricted Domain, we get to choose our example. Let

G_1 : x > y > z
G_2 : y > z > x
H \ G : z > x > y

G is decisive so $\overline{D}_G(y,z)$ so y P z .

Case 1: x P z

Then G_1 is decisive by the Field Expansion Lemma and Independence of Irrelevant Alternatives.

Case 2: z P x

transitivity implies y P x

Field Expansion Lemma & Independence of Irrelevant Alternatives implies G_2 is decisive. QED

<u>Proof of the Arrow Possibility Theorem:</u> Pareto Principle implies that H is decisive. Group contraction lemma implies that we can successively eliminate elements of H so that remaining subsets are still decisive. Repeat. Then there is $j \in H$ so that $\{j\}$ is decisive. Then j is a dictator. QED