Nonparametric Adaptive Learning with Feedback

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1. INTRODUCTION

Many economic dynamic systems involving learning can exhibit feedback because current learning will Granger-cause current and future realizations of state variables, affecting the evolution of the dynamic system. A popular method that allows econometricians to handle such feedbacks is the Kalman filter technique, which turns out to be a special case of parametric stochastic approximation (SA) procedures with feedback. These are recursive methods that can generally be used to locate the roots of an "unknown" mapping $\overline{M}$, or the extreme points of an "unknown" objective function, where the unknowns may be observable up to some error. A typical SA algorithm is the following Robbins-Monro (1951) procedure: choose vectors $\hat{\theta}_0, \hat{\xi}_0$ independently, and update according to

$$\hat{\theta}_{n+1} = \hat{\theta}_n + a_n M_n(\hat{\xi}_n, \hat{\theta}_n) \equiv \hat{\theta}_n + a_n [\, \overline{M}(\hat{\theta}_n) + U_n(\hat{\xi}_n, \hat{\theta}_n) \,], \quad \hat{\xi}_{n+1} = R_n(\hat{\xi}^n, \hat{\theta}^{n+1}, Z_{n+1}),$$

where U_n is the error, $\{ Z_n \}$ is an exogeneous random noise sequence observable or not, $\hat{\xi}^n \equiv (\hat{\xi}_0, \ldots, \hat{\xi}_n)$, and $\hat{\theta}^{n+1} \equiv (\hat{\theta}_0, \ldots, \hat{\theta}_{n+1})$. In the economic learning context, $\hat{\theta}_{t+1}$ represents "knowledge" at time n+1, while $\hat{\xi}_{n+1}$ may represent "action" at time n+1. When for each $n = 0,1,2,\ldots$, $R_n(\xi^n, \cdot, z_{n+1})$ is a constant function for each ξ^n and z_{n+1}, then we call the algorithm a SA procedure *without feedback*. Otherwise we have a SA procedure *with feedback*. Note that a special case of feedback can be rewritten as a procedure without feedback, i.e., when $\hat{\xi}_n \equiv f_n(\hat{\theta}_n, Z_n)$ for all $n \geq 0$, where f_n is a known function and Z_n is observable. One can always rewrite a SA procedure without feedback as $\hat{\theta}_{n+1} = \hat{\theta}_n + a_n [\, \overline{M}(\hat{\theta}_n) + F_n(W_n, \hat{\theta}_n) \,]$, where $\{ W_n \}$ is an observable exogeneous stochastic process. Within the class of SA procedures without feedback, if F_n is only a function of W_n, we have so-called "$\theta -$ *independent* errors"; otherwise, we have "$\theta -$ *dependent* errors".

A parametric SA procedure is easy to compute and has a nice interpretation as a deterministic dynamic system with error. This error term has a long-run effect which, on average, is negligible. In particular, the procedure allows feedback within the dynamic system, permits non-linearity of the mapping $\overline{M}$, and allows non-stationarity and serial correlation of the innovation process $\{ Z_n \}$. Some parametric SA procedures have been used in econometrics as tools for parametric recursive GMM - estimation, estimation with dynamic latent variables, and forecasting. Recently some economists in microeconomics and macroeconomics have applied parametric SA procedures with feedback to such problems as modeling adaptive learning with possibly "bounded rationality", selecting a Nash equilibrium without imposing complete "common knowledge", solving stochastic dynamic programming problems or systems of nonlinear stochastic equations, and for disequilibrium dynamic analysis.

Nevertheless, economics is a domain in which economic agents consciously or unconsciously interact with a random environment evolving over time in a complicated way. Parametric models in economics can capture only relatively simple (e.g., linear) relationships and thus may be a bad approximation to the true economic relationships. Policy analysis based on poor parametric models might lead to very misleading conclusions, and theoretical results might be very sensitive to the specific parametric forms. We can avoid these problems by permitting $\hat{\theta}_n$ to take values in a general function space, thereby obtaining nonparametric procedures that can capture nonlinear stochastic dynamic systems plausible in economics. This motivates us to investigate properties of some Banach and Hilbert space-valued stochastic approximation ($\mathbb{B}$ or $\mathbb{H}-$SA) procedures and their applications to adaptive learning by economic agents using nonparametric recursive GMM - estimation.

Mathematicians and engineers have established results for such asymptotic properties as convergence, asymptotic normality and rate of convergence for $\mathbb{B}$ and $\mathbb{H}-$SA algorithms without feedback. Most of these algorithms are not directly computable, and asymptotic results are valid only for martingale difference error (U_n) cases. A recent substantive advance in this literature has been made by Yin (1992), who analyzed a projection-based $\mathbb{H}-$SA algorithm. He allows the error U_n to exhibit greater time dependence than do martingale differences. To the best of our knowledge, Yin's and all other current results consider only the $\theta-$independent error case without feedback. In contrast, Chen and White (1992) propose a sieve-type $\mathbb{H}-$SA procedure (without feedback) with $\theta-$dependent errors. In particular, this is a procedure with finite-dimensional projection and random truncation. It is easy to compute, allows non-linearity, imposes no prior bound on the value of the estimator, and permits Hilbert space-valued mixingale innovations. We obtain asymptotic properties including almost-sure convergence in norm, asymptotic normality, law of iterated logarithm and mean rate of convergence. One can apply these results to nonparametric recursive GMM - estimation and agent learning.

An open problem from our previous paper is to explicitly allow the possibility of feedback within the dynamic system. In this paper, we modify our previous framework to incorporate feedback. Section 2 gives several important economic examples to motivate our research. Section 3 studies a Robbins-Monro procedure with feedback (RMF) in a Banach space, and its almost-sure convergence. We prove results for almost-sure convergence in a certain topology under conditions similar to the weakest possible conditions for multivariate RMF (KC, 1978). Section 4 specializes to several RMFs in a Hilbert space, and obtains their almost-sure convergence in the weak topology. Sufficient conditions are provided in section 5 to allow the innovation process $\{ Z_n \}$ to be a Hilbert space-valued near epoch dependent function of mixing processes. All the results in sections 3, 4 and 5 are new. In section 6, we apply our general theorems to establish convergence results for the economic examples in section 2. Section 7 is a brief summary. Some definitions and all proofs are collected into a mathematical appendix.

2. EXAMPLES

Recently macroeconomists (e.g., Marcet & Sargent, Woodford) and game theorists (e.g., Fudenberg & Kreps, Crawford) have considered the consequences of dropping the mutual consistency assumption in the concept of rational expectations equilibrium (REE) or Nash equilibrium (NE), and instead search for plausible agent learning behaviors that lead to a "sensible" REE or NE. Many of these learning models are parametric stochastic approximation algorithms with feedback. However, parametric learning can lead to incorrect belief equilibria as shown by Kuan & White (KW) (1994). The following examples illustrate that it may be more natural to learn the true functional relationships nonparametrically.

Example 2.1: (Generalization of Bray’s Model)

Bray’s (1983) model is one of the first adaptive learning models in the rational expectations literature. The following is a simple version of Bray’s story. Bray assumes the equilibrium price p_t for a single commodity is determined by the market-clearing condition $p_t = a + b\, p^e{}_{t+1} + u_t$, where $0 < b < 1$, a is a constant, $p^e{}_{t+1}$ is the price that market participants expect to prevail at time $t+1$ given past price information, and $\{ u_t \}$ is an exogenous process independent identically distributed with $E[\, u_t \,] = 0$ and $E[\, u_t^2 \,] < \infty$. Bray shows that there is a unique rational expectations equilibrium

$p_t = \theta_o + u_t$, where $\theta_o \equiv a / [1-b]$.

Next, Bray supposes that homogeneous agents observe data $\{ p_t \}$ and estimate p_t^e according to $p^e{}_{t+1} \equiv \hat{\theta}_t = \hat{\theta}_{t-1} + t^{-1} [p_{t-1} - \hat{\theta}_{t-1}]$, $t \geq 1$, where $\hat{\theta}_0$ is an arbitrary random variable. Note that the market-clearing price process now becomes $p_t(\hat{\theta}_t) = a + b\,\hat{\theta}_t + u_t$. Bray shows that $\{ p_t(\hat{\theta}_t) \}$ converges almost surely to the unique rational expectations equilibrium, $\theta_o + u_t$ (i.e., $p_t(\hat{\theta}_t) - \theta_o - u_t \to 0$ $a.s. - I\!P$ as $t \to \infty$).

Now suppose that prices are affected by fundamentals x exogenous to this market. This can be handled by generalizing Bray's (1983) model to a function space. We now assume the equilibrium price function $p(x, t)$ for a single commodity is determined by the market-clearing condition: $p(x, t) = a(x) + b\, p^e(x, t+1) + u_t$, where $x \in [-1,1]$ is a measure of market fundamentals, $a(\cdot)$ is a measurable function with $\int_{[-1,1]} [a(x)]^2\, dx < \infty$ (so $a(\cdot) \in L_2[-1,1]$, a separable Hilbert space -- it suffices that a is continuous on $[-1,1]$), $0 < b < 1$, $p^e(x, t+1)$ is the price that market participants expect to prevail at time $t+1$ when x occurs given past price and fundamental information, and $\{ u_t \}$ is an *i.i.d.* unobservable exogenous random sequence with compact support, zero mean, and finite variance. It can be shown that there is a mean price function as the unique strictly stationary solution $p(x, t) = \theta_o(x) + u_t$, where $\theta_o(x) \equiv a(x) / [1-b]$.

Suppose that homogeneous agents can observe data on prices and fundamentals X_t, where $\{ X_t \}$ is a possibly time dependent exogenous random sequence with support $[-1,1]$. By analogy with Bray's approach, suppose agents estimate p^e as $p^e(x, t+1) = \hat{\theta}_t(x)$, for $x \in [-1,1]$, $\hat{\theta}_t$ a random continuous function on $[-1,1]$. Note that the market-clearing price process now becomes $p(x, \hat{\theta}, t) = a(x) + b\,\hat{\theta}_t(x) + u_t$. Suppose that at the beginning of time t, agents form an estimate as $\hat{\theta}_t(x)$, and then observe X_t and $\hat{p}_t$, where $\hat{p}_t = a(X_t) + b\,\hat{\theta}_t(X_t) + u_t$, with agents using one of the following methods to estimate $\hat{\theta}_t(x)$:

Method 1:

Let $K : I\!R \to I\!R$ be any density function. Let $\{ h_t \}$ be a sequence of positive numbers decreasing to zero as $t \to \infty$, the "bandwidth". The agents set $\hat{\theta}_0 \equiv 0$ and for $t \geq 0$,

$$\hat\theta_{t+1}(x) = \hat\theta_t(x) + (t+1)^{-1} K(\,(x - X_t)/h_t\,)\,[\,\hat p_t - \hat\theta_t(x)\,]/h_t\,, \qquad x \in [-1,1]\,.$$

Method 2:

Let $K : \mathbb{R} \to \mathbb{R}$ be a kernel density function. Let $\{ h_t \}$ be a sequence of positive numbers decreasing to zero as $t \to \infty$. The agents pick $\hat\theta_0$ arbitrarily from $L_2[-1,1]$, and for $t \ge 0$ set

$$\hat\theta_{t+1}(x) = \hat\theta_t(x) + (t+1)^{-1}\,[\,h_t^{-1} K(\,(x - X_t)/h_t\,)\,\hat p_t\,[\,\hat f_t(x)\,]^{-1} - \hat\theta_t(x)\,]\,, \qquad x \in [-1,1]\,,$$

$$\hat f_{t+1}(x) = \hat f_t(x) + (t+1)^{-1}\,[\,h_t^{-1} K(\,(x - X_t)/h_t\,) - \hat f_t(x)\,]\,, \qquad x \in [-1,1]\,.$$

Method 3:

Let K, h_t and $\hat f_t$ be as in Method 2. The agents set $\hat\theta_t = \hat m_t / \hat f_t$, where

$$\hat m_{t+1}(x) = \hat m_t(x) + (t+1)^{-1}\,[\,h_t^{-1} K(\,(x - X_t)/h_t\,)\,\hat p_t - \hat m_t(x)\,]\,, \qquad x \in [-1,1]\,.$$

Method 4:

Let $\{ e_l(\,\cdot\,) \}$ be a complete orthonormal basis for $L_2\,[-1,1]$, say $e_l(x) = 2^{1/2} \sin(\,l\,x\,\pi\,)$ for $x \in [-1,1]$. The agents again set $\hat\theta_t = \hat m_t / \hat f_t$ while now $\hat m_t$ and $\hat f_t$ are recursive series estimators:

$$\hat m_{t+1}(x) = \hat m_t(x) + (t+1)^{-1}\,[\,\Sigma_{1 \le j \le k(t)}\, e_j(X_t)\, e_j(x)\,\hat p_t - \hat m_t(x)\,]\,, \qquad x \in [-1,1]\,,$$

$$\hat f_{t+1}(x) = \hat f_t(x) + (t+1)^{-1}\,[\,\Sigma_{1 \le j \le k(t)}\, e_j(X_t)\, e_j(x) - \hat f_t(x)\,]\,, \qquad x \in [-1,1]\,.$$

We now can ask the key question: Does $\{ p(\,\cdot\,, \hat\theta_t\,, t\,) \}$ converge in any meaningful way to the unique solution $\theta_o(\,\cdot\,) + u_t$? The theory of Sections 3, 4 and 5 applies to give precise and general conditions under which an affirmative answer can be given. □

Example 2.2: (Generalization of Arthur's Learning Algorithms)

Arthur (1990) has studied the following type of learning automaton (also known as urn process, classifier system, or stochastic replicator dynamic):

A single agent can undertake one action from { 1, 2,..., N}, $N < \infty$ at each time. At time t, the simple minded agent associates a vector of strengths $S_t = (\,S_t(1)\,, \ldots, S_t(N)\,)^T$ with the actions 1,2,..., N, and updates his probabilities $\theta_t = (\,\theta_t(1)\,, \ldots, \theta_t(N)\,)^T$ of taking each action on the basis of the realization of random payoffs he has experienced during the past. He then chooses his action randomly from

$\{ 1, 2,..., N \}$ according to the probability θ_t. At the end of time t (or the beginning of time $t+1$), the agent observes the realization of random payoffs $Q_t(j)$ if action j is actually chosen, and updates the strength according to

$$0 < S_0(i) < \infty \text{ for all } i , \quad S_{t+1}(i) = S_t(i) + \beta_t(i, \theta_t) \text{ for } t \geq 0 ,$$

where $\beta_t(i,\theta) = Q_t(i)$ if i is actually chosen at time t (i.e., $\beta_t(i,\theta_t) = Q_t(i)$ with probability $\theta_t(i)$), $\beta_t(i,\theta) = 0$ if i is not chosen at time t. Let $C_t = \sum_{j=1}^{N} S_t(j)$. Define $\theta_t = S_t / C_t$ and $a_t = (C_{t+1})^{-1}$.

The evolution of the probability of choosing action i is

$$\theta_{t+1}(i) = \theta_t(i) + a_t \, [\beta_t(i, \theta_t) - \theta_t(i) \sum_{j=1}^{N} \beta_t(j, \theta_t)] , \quad i = 1, ..., N .$$

Arthur (1990) assumes that $\{ Q_t \}$ is a sequence of independent, identically distributed random payoffs which is independent of θ_t. When $a_t = O (t^{-1}) \;\; a.s. - I\!P$, Arthur (1990) shows that $\{ \theta_t \}$ converges almost surely to the optimal payoff action vertex of the simplex of probabilities Δ^N; i.e., the limit of $\{ \theta_t \}$ is the solution to $\theta(i) \, [E[Q_t(i)] - \sum_{j=1}^{N} E[Q_t(j)] \theta(j)] = 0$ for all $i = 1,2,..., N$.

There are many variations of the above learning algorithm. For example, Roth and Erev (1993) have utilized such algorithms to study some experimental data in a game theoretical framework. We modify Arthur’s model by allowing the random payoff to be dependent and heterogeneous and by letting the action space be a compact subset of real line, so that instead of being a vector of length $N < \infty$, θ_t now belongs to a function space. For concreteness, we consider the following learning model: Suppose a single agent has an action set [0,1]. At the beginning of time t, he chooses a density $\theta_t(\cdot)$ on $[0,1]$ and then chooses an action randomly from $[0,1]$ according to $\theta_t(\cdot)$. He then observes action x_t occurr and receives a positive random payoff $Q (x_t , Z_t)$, where $Q(\cdot , \cdot)$ is a unknown function and Z_t is a unobservable, exogeneous, possibly time dependent, Hilbert-valued random element. At time t, the agent associates strength $S_t(x)$ with any possible action $x \in [0,1]$, and updates in the following way:

$$0 < S_0(x) < \infty \text{ for } x \in [0,1] , \; 0 < \int_{[0,1]} S_0(x) \, dx < \infty , \quad S_{t+1}(x) = S_t(x) + \beta_t (x, x_t, Z_t, \theta_t) ,$$

$$\text{with } \beta_t(x, x_t, Z_t, \theta_t) = Q(x_t, Z_t) (h_t)^{-1} K((x - x_t) / h_t) \text{ if } x_t \text{ is actually chosen ;}$$

$\beta_t(x, x_t, Z_t, \theta_t) = 0$ otherwise. Here again $K(\cdot)$ is a kernel, and $\{h_t\}$ is a sequence of small positive numbers chosen by the agent. Define $\theta_t(x) = S_t(x) / \int_{[0,1]} S_t(y)\, dy$ and $a_t = [\int_{[0,1]} S_{t+1}(y)\, dy]^{-1}$. Then for any $x \in [0,1]$,

$$\theta_{t+1}(x) = \theta_t(x) + a_t\, [\beta_t(x, x_t, Z_t, \theta_t) - \theta_t(x) \int_{[0,1]} \beta_t(y, x_t, Z_t, \theta_t)\, dy\,] .$$

Our theory in the next section permits development of reasonable conditions under which this agent will learn the optimal payoff action in an appropriate sense. □

Example 2.3: (Suggested by Michael Woodford)

A representative agent seeks to maximize $E\,[\sum_{t=0}^{\infty} \beta^t\, U(c_t)\,]$, where $0 < \beta < 1$, and $U(c)$ is an increasing concave function. The budget constraint each period is: $c_t + s_t \le r_t\, s_{t-1}$, $c_t \ge 0$, $s_t \ge 0$, where r_t is gross real return on savings at period t with $r_t > 0$ $a.s. - I\!P$, s_t is saving and c_t consumption at period t. The value of r_t is observed before c_t, s_t are chosen. Finally s_{-1} is given as an initial condition, and $\{r_t\}$ is an $i.i.d.$ sequence with unknown distribution.

The optimal behavior (with correct expectation) is: at period t, choose c_t, s_t to maximize $U(c_t) + W(s_t)$ subject to $c_t + s_t \le r_t\, s_{t-1}$, $c_t \ge 0$, $s_t \ge 0$, where $W(s)$ is the expected value of savings, which in turn satisfies

$$W(s) = E_t[\beta\, (\max_{c_t, s_t} [\, U(c_t) + W(s_t)\,]\,)\,] \quad \text{subject to} \quad c_t + s_t \le r_t\, s\,, \;\; c_t \ge 0\,, \;\; s_t \ge 0\,,$$

where $E_t[\cdot]$ is taken conditionally on r_t.

Suppose that the agent has to learn the function $W(s)$. Then at each period, the agent can choose c_t, s_t to maximize $U(c_t) + W_t^e(s_t)$ subject to $c_t + s_t \le r_t\, s_{t-1}$, $c_t \ge 0$, $s_t \ge 0$, where $W_t^e(s)$ is the belief at date t about the function $W(s)$, and $W_0^e(\cdot)$ is chosen to be an arbitrary increasing concave function, e.g. $U(\cdot)$. The agent computes the function

$$\hat{W}_t(s) = \beta\, (\max_{c_t, s_t} [\, U(c_t) + W_t^e(s_t)\,]\,) \quad \text{subject to} \quad c_t + s_t \le r_t\, s\,, \;\; c_t \ge 0\,, \;\; s_t \ge 0$$

and updates his belief according to $W_{t+1}^e(s) = W_t^e(s) + a_t\, [\,\hat{W}_t(s) - W_t^e(s)\,]$ for a sequence of decreasing weights $\{a_t\}$. Under what conditions can one show that $W_t^e \to W$? The results of sections 3 and 4 apply to provide a satisfying answer to this question. □

Example 2.4: (Generalization of Fictitious Play)

Fictitious play (Brown, 1951; Robinson, 1951) belongs to a class of famous models of learning and behavior in game theory. A typical fictitious play model assumes two players, two strategies, simultaneous moves, and at the beginning of each period, each player uses the historical frequency of the other player's past actions to form a belief about the other player's possible strategy. The player then chooses his own strategy to maximize his current period's expected payoff given his belief. Fudenberg and Kreps (1993) have considered an augmented game of two-player, two-strategy fictitious play, consisting of the original game with both players' payoffs subject to *i.i.d.* shocks, and each player knowing only his own payoff shocks. Fudenberg and Kreps (1993) have applied convergence results of parametric stochastic approximation to show that learning players will converge to the unique Nash Equilibrium both in beliefs and in strategies.

In this example, we consider a generalization of fictitious play with countinuum strategies. As an illustration, we assume that the game has the following structure: two players, simultaneous moves, infinite repetition, and each player's action space is $[0,1]$. At the end of period t, player i observes both players' actual actions $\xi_{1,t}$ and $\xi_{2,t}$, forms a belief $\theta_{ij,t+1}$ about player j's action $\xi_{j,t+1}$ in the next period, where the belief $\theta_{ij,t+1}$ is a probability density over player j's action space $[0,1]$. Player i then decides his action for the next period $\xi_{i,t+1}$ according to his response function $R_{i,t}$ which is derived from maximizing his current period's expected payoff given his beliefs. This behavior is taken to be myopic as in Fudenberg and Kreps (1993). In particular, player i will choose $\xi_{i,t+1} \in [0,1]$ to maximize

$$\int_{[0,1]} Q_{i,t+1}(\xi_{i,t+1}, Z_{i,t+1}, \xi_j)\,\theta_{ij,t+1}(\xi_j)\, d\xi_j ,$$

where $Q_{i,t+1}$ is concave and differentiable in $\xi_{i,t+1}$, and $Z_{i,t+1}$ is a payoff shock to player i known at time $t+1$, which is unknown to the other player for all time. In contrast to Fudenberg and Kreps (1993), who take $\{ Z_{i,t+1} \}$ to be an exogenous *i.i.d.* scalar sequence, we take $\{ Z_{i,t+1} \}$ to be an exogenous sequence of dependent Hilbert-valued random elements.

Given his belief $\hat{\theta}_{ij,t+1}$, player i's best action at time $t+1$, $\hat{\xi}_{i,t+1} = R_{i,t}(\hat{\theta}_{ij,t+1}, Z_{i,t+1})$, is a solution to

$$\int_{[0,1]} d/d\xi_i \,[\, Q_{i,t+1}(\hat{\xi}_{i,t+1}, Z_{i,t+1}, \xi_j)\,]\,\hat{\theta}_{ij,t+1}(\xi_j)\, d\xi_j = 0 .$$

For example, if the payoff functions are quadratic,

$$Q_{i,t+1}(\xi_{i,t+1}, Z_{i,t+1}, \xi_{j,t+1}) = -(\xi_{i,t+1} + Z_{i,t+1} - \xi_{j,t+1})^2 ,$$

then $\hat{\xi}_{i,t+1} = \int_{[0,1]} x\,\hat{\theta}_{ij,t+1}(x)\,dx - Z_{i,t+1}$. Suppose player i forms his beliefs as

$$\hat{\theta}_{ij,t+1}(x) = \hat{\theta}_{ij,t}(x) + (t+1)^{-1}\,[\,(h_{i,t})^{-1} K_i((x-\hat{\xi}_{j,t})/h_{i,t}) - \hat{\theta}_{ij,t}(x)\,], \quad x \in [0,1] .$$

Again, our theory will provide sufficient conditions for the convergence to Nash Equilibrium. □

All the preceeding examples can be reduced to the convergence problem of the following algorithm:

$$\hat{\theta}_{t+1} = \hat{\theta}_t + a_t M_t(\hat{\xi}_t, \hat{\theta}_t), \quad \hat{\xi}_{t+1} = R_t(\hat{\xi}^t, \hat{\theta}^{t+1}, Z_{t+1}),$$

where a_t could be random, but all a_t are of the order of $1/t$. In all examples, $\hat{\theta}_t$ is a function being learned; $\hat{\xi}_t$ is a function in some examples and is a vector in other examples, often corresponding to agent actions and governing the evolution of the dynamics; and Z_t is an exogenous random element.

In this paper, we provide nonparametric SA algorithms which permit feedback, nonlinearity of the updating function, and time dependent noise. In particular, we put our SA with feedback algorithms in a separable Banach or Hilbert space. Convergence results for Banach space algorithms require satisfaction of somewhat abstract memory conditions. These conditions can be easily verified if we have independent, mixing or martingale difference noise processes, but there are presently no convenient mathematical theorems available to deal with Banach space-valued highly time dependent, heterogeneous random processes analogous to mixingales. Accordingly, we use separable Hilbert spaces to accomodate the highly time dependent, heterogeneous noise processes that unavoidably arise in the presence of feedback. Although we are motivated by the desire to provide algorithms and results for the purpose of modeling adaptive learning behaviors in decision theory and game theory, a further benefit of our analysis is to provide nonparametric recursive estimators in time series econometrics when the data exhibit certain plausible Granger-causality structure.

3. BASIC ALGORITHM AND ALMOST-SURE CONVERGENCE IN A BANACH SPACE

We begin by describing the data generating process.

ASSUMPTION A.1: $(\Omega, F, I\!P)$ is a complete probability space on which is defined an (exogenous) stochastic process $\{ Z_n : \Omega \to G \,;\, n = 0,1,2,... \}$ (i.e., a sequence of $F / B(G)$ –measurable mappings, generated by nature), where G is a bounded subset of a real separable Banach space.

Next we describe the learning update functions.

ASSUMPTION A.2: (a) let Ξ be a compact convex subset of a real separable Banach space under metric r_Ξ, and let K be a compact convex subset (containing the origin) of a real separable Banach space under translation invariant metric r_K, metrizing a topology such that for all K –valued cadlag (right continuous with left limits) functions $f_n, f : I\!R \to K$, the integral below is well defined for all t such that the integral remains in K, and the following implication holds (where the convergence is in sup-norm on any bounded intervals): $f_n \to f$ implies $\int_{[0,t]} f_n(s)\, ds \to \int_{[0,t]} f(s)\, ds$ (e.g., norm or weak topology). (b) $M_n : \Xi \times K \to K$ is continuous for each n, and $co\, (\cup_{n \le j \le l} M_j(\Xi \times K) \,;\, 0 \le l - n < \infty)$ (the convex hull of $\cup_{n \le j \le l} M_j(\Xi \times K) \,;\, 0 \le l - n < \infty)$ is bounded $(- r_K)$ for all n large enough.

Note that the metric r_K is only assumed to be translation invariant (i.e., $r_K(x+z, y+z) \equiv r_K(x, y)$ for all $x, y, z \in K$), but not necessarily scalar invariant (i.e., $r_K(cx, cy) \equiv |c|\, r_K(x, y)$ for all $c \in I\!R$, $x, y \in K$). For example, we can take $r_K(x, y) = \sum_{j=1}^{\infty} 2^{-j} p_j(x-y) / [1 + p_j(x-y)]$ for any $x, y \in K$, where $\{ p_j \}$ is a sequence of seminorms on K such that $\cap_{1 \le j \le \infty} \{ x : p_j(x) = 0 \} = \{ 0 \}$.

The learning rate is described by the next assumption.

ASSUMPTION A.3: $\{ a_n \,;\, n = 0,1,2,... \}$ is a sequence of nonincreasing positive random numbers such that: $a_n \to 0$ as $n \to \infty$ $a.s.-I\!P$ and $\sum_{n=0}^{\infty} a_n = \infty$ $a.s.-I\!P$.

We now describe the underlying law of motion of the dynamic system.

ASSUMPTION A.4: For each n, $R_n : \Xi^n \times K^{n+1} \times G \to \Xi$ is a known bounded Borel measurable function, where $\Xi^n \equiv \times_{t=0}^{n} \Xi$ and $K^{n+1} \equiv \times_{t=0}^{n+1} K$.

The next assumption specifies initial values for the learning recursions.

ASSUMPTION A.5: $\hat{\xi}_0 : \Omega \to \Xi$ and $\hat{\theta}_0 : \Omega \to K$ are arbitrary measurable mappings independent of $\{ Z_n \}$.

We now have sufficient structure to define the learning recursions of interest:

$$\hat{\theta}_{n+1} = \hat{\theta}_n + a_n M_n(\hat{\xi}_n, \hat{\theta}_n), \quad \hat{\xi}_{n+1} = R_n(\hat{\xi}^n, \hat{\theta}^{n+1}, Z_{n+1}), \quad n = 0,1,2,...,$$

with $\hat{\xi}^n = (\hat{\xi}_0, ..., \hat{\xi}_n)$, $\hat{\theta}^{n+1} = (\hat{\theta}_0, ..., \hat{\theta}_{n+1})$.

The above learning recursion is simply a Banach space ($I\!B$) version of the Robbins-Monro procedure with feedback (RMF) introduced by KC (1978). Berman and Shwartz (1989) (BS) present a Banach space-valued Robbin-Monro (RM) procedure without feedback, in which $M_n : G \times \Xi \to \Xi$ is the form $M_n(Z_n, \hat{\theta}_n) = M(\hat{\theta}_n) + Z_n$ for all n. Given KC's (1978) approach to the real-valued RMF, and BS's (1989) approach to the Banach-valued RM, it is plausible that the essential steps to establish almost sure convergence (in a certain topology) for a Banach-valued RMF are first to find a suitable compact convex metric space (K, r_K), and second to show (a) $\hat{\theta}_n \in K$ for all n, and (b) for any $\varepsilon > 0$ and a suitable function $\overline{M}$,

$$\lim_{n\to\infty} I\!P \,[\, \sup_{j \ge n} \max_{t \le T} r_K(\Sigma_{m(jT) \le i \le m(jT+t)-1}\, a_i \,[\, M_i(\hat{\xi}_i, \hat{\theta}_i) - \overline{M}(\hat{\theta}_i) \,]\,, 0) \ge \varepsilon \,] = 0 \ ,$$

where $t_0 = 0$, $t_n = \Sigma_{0 \le i \le n-1}\, a_i$ and $m(t) \equiv \max [\, n > 0 : t_n \le t \,]$ if $t \ge 0$ and $m(t) \equiv 0$ if $t < 0$. We formalize this requirement as

ASSUMPTION A.6: There is a function $\overline{M} : K \to K$ such that: (a) $\overline{M}$ is continuous $(-\, r_K)$; and (b) for each $0 < T < \infty$, each $\theta \in K$ and any $\varepsilon > 0$, the following holds:

$$\lim_n I\!P\ [\ \sup_{j\ge n} \max_{t\le T} r_K\ (\ \Sigma_{m(jT)\le i\le m(jT+t)-1}\ a_i\ [\ M_i(\xi_i(\theta)\,,\theta\,)-\overline{M}(\theta)\ ]\ ,\ 0\ )\ge\varepsilon\ ]=0\ ,$$

where $\xi_n(\theta)$ is defined recursively as

$$\xi_{n+1}(\theta)=R_n\ (\xi^n(\theta)\,,\theta^{n+1}\,,Z_{n+1}\,)\ ,\quad n=0,1,2,...\ ;\ \ \xi_0(\theta)=\xi^0(\theta)=\hat{\xi}_0\ ,$$

and $\theta^{n+1}\equiv(\theta\,,\ \cdots\ ,\theta\,)$, the point in K^{n+1} with identical coordinate θ in each position.

Let μ denote a sequence $\{\mu_n\}$ with $\mu_n\in K$ for each n. Define the sequence $\{\xi_n(\mu)\}$ recursively as

$$\xi_{n+1}(\mu)=R_n\ (\xi^n(\mu)\,,\mu_i\,,\ 0\le i\le n+1\,,\ Z_{n+1}\,)\ ,\ \ n=0,1,2,...\ ;\ \ \xi_0(\mu)=\xi^0(\mu)=\hat{\xi}_0\ .$$

Let $\overline{\mu}(\cdot)$ be the function with value μ_i on $[\,t_i\,,t_{i+1}\,)$. Let $\overline{\xi}^{\,o}(\mu\,)$ be the piecewise right continuous constant interpolation of $\{\xi_n(\mu\,)\}$ on $[\,0\,,\infty\,)$ with interpolation intervals $\{\,a_n\,\}$, i.e., $\overline{\xi}^{\,o}(\mu\,,t\,)=\xi_n(\mu\,)$ for $t\in[\,t_n\,,t_{n+1}\,)$.

ASSUMPTION A.7: There exists a $I\!P$ –null set Ω_o such that for any $\omega\in\Omega-\Omega_o$, for each real $q_1>0\,,q_2>0$, we have: for each $\varepsilon>0$, there exists $\delta>0$ such that for any infinite subsequence $\{\,n'\,\}$,

$$\text{(1)}\ \lim\sup_{n'}\ \sup_{-q_2\le s,\tau\le q_1} r_K\ (\overline{\mu}(\,t_{n'}+s\,)\,,\overline{\mu}(\,t_{n'}+\tau\,)\,)\le\delta\quad\text{and}$$

$$\text{(2)}\ \lim\sup_{n'}\ \sup_{-q_2\le s\le q_1} r_K\ (\overline{\nu}(\,t_{n'}+s\,)\,,\overline{\mu}(\,t_{n'}+s\,)\,)\le\delta\,,$$

then for that subsequence $\{\,n'\,\}$,

$$\text{(3)}\ \lim\sup_{n'}\ \sup_{0\le s\le q_1} r_\Xi(\,\overline{\xi}^{\,o}(\mu\,,t_{n'}+s\,)\,,\overline{\xi}^{\,o}(\nu\,,t_{n'}+s\,)\,)\le\varepsilon\ .$$

Here both $\mu\equiv\{\mu_n\}$ and $\nu\equiv\{\nu_n\}$ can be either sequences of constant values θ and θ' or one is constant and the other is the sequence of sample values of $\{\,\hat{\theta}_n(\omega)\,\}$.

Notice we can rewrite Assumption A.7 in a non-interpolated form with the following versions of (1) - (3):

$$\text{(1)}\ \lim\sup_{n'}\ \sup_{-q_2\le n_1,n_2\le q_1} r_K\ (\mu_{n'+n_1}\,,\mu_{n'+n_2}\,)\le\delta\,,$$

$$\text{(2)}\ \lim\sup_{n'}\ \sup_{-q_2\le n_1\le q_1} r_K\ (\nu_{n'+n_1}\,,\mu_{n'+n_2}\,)\le\delta\,,$$

$$\text{(3)}\ \lim\sup_{n'}\ \sup_{0\le n_3\le q_1} r_\Xi(\xi_{n'+n_3}(\mu\,)\,,\xi_{n'+n_3}(\nu\,)\,)\le\varepsilon\ .$$

Let $\theta^o(\cdot)$ be a piecewise linear interpolation of $\{ \hat{\theta}_n \}$ with interpolation intervals $\{ a_n \}$, i.e.,

$$\theta^o(t) \equiv \hat{\theta}_n \times (t_{n+1} - t) / a_n + \hat{\theta}_{n+1} \times (t - t_n) / a_n \quad \text{for } t \in [t_n, t_{n+1}) .$$

Define $\theta^n(\cdot)$, the left shift of $\theta^o(\cdot)$, by $\theta^n(t) = \theta^o(t_n + t)$.

THEOREM 3.1: Given A.1 - A.7 for the Banach-valued RMF, if $\hat{\theta}_n \in K$ for all n $a.s.-\mathbb{P}$, then:

(i) $\{ \theta^n(\cdot) \}$ is uniformly bounded and equicontinuous on bounded time intervals almost surely in the r_K–metric.

(ii) $\{ \theta^n(\cdot) \}$ is sequentially compact in (K , r_K); and any r_K–limit θ satisfies the ODE $\dot{\theta} = \overline{M}(\theta(\cdot))$, where $\dot{\theta}(t) = d\theta(t) / dt$.

Let Θ^* be the set of locally asymptotically stable (in the sense of Liapunov) equilibria in (K , r_K) for the above ODE with domain of attraction $da\,(\Theta^*) \subset K$. Fix a compact subset $C \subset da(\Theta^*)$.

(iii) If $\hat{\theta}_n \in C$ *infinitely often*, then $\hat{\theta}_n \to \Theta^*$ as $n \to \infty$ $a.s.-\mathbb{P}$ in the r_K–metric, i.e., there exists Ω^* with $\mathbb{P}(\Omega^*) = 1$ such that for any $\omega \in \Omega^*$, $\lim_n [\inf_{\theta \in \Theta^*} r_K (\hat{\theta}_n(\omega) , \theta)] = 0$.

To illustrate the content of this result, we now restrict K and Ξ to be bounded closed balls of real separable reflexive Banach spaces (e.g., Hilbert spaces, l_p–spaces, L_p–spaces of integrable functions, Sobolev spaces with generalized L_p derivatives, $1 < p < \infty$), and obtain almost-sure convergence in the weak topology. Recall that the weak topology of a Banach space ($\mathbb{B}$) is the topology obtained by taking as base all sets of the form $n\,(x;A,\varepsilon) = \cap_{h \in A} \{ y \in \mathbb{B} : | h (y - x) | < \varepsilon \}$, where $x \in \mathbb{B}$, A is a finite subset of $\mathbb{B}^*$, the space of continuous linear functionals on $\mathbb{B}$, and $\varepsilon > 0$. The weak topology of a bounded closed set of a separable reflexive Banach space is metrizable; we denote d and d_Ξ the metrics which specify the weak topologies on K and Ξ respectively. For example, we can define $d\,(x,y) = \sum_{n=1}^{\infty} 2^{-n} | h_n(x) - h_n(y) | / (1 + | h_n(x) - h_n(y) |)$, where $\{ h_n \}$ is a countable dense subset of $\mathbb{B}^*$. Under the weak topology, any norm-bounded subset of a reflexive Banach space is compact; Assumption A.2(a) is automatically satisfied; If for each n, M_n is continuous under the weak topology (i.e., weakly sequentially continuous), then $M_n(\Xi \times K)$ is compact, and $\cup_{n \le j \le l} M_j(\Xi \times K)$ is compact as long as $l - n < \infty$. Hence $co\,(\cup_{n \le j \le l} M_j(\,\Xi \times K\,)\,) :\ 0 \le l - n < \infty\,)$ is compact, thus bounded. Therefore we obtain the following almost-sure convergence in weak topology as an application of Theorem 3.1.

COROLLARY 3.2: Let K and Ξ be norm-bounded closed balls of real separable reflexive Banach spaces with the weak topology-induced metrics d and d_Ξ respectively. Suppose the following conditions hold: (1) A.1, A.3 - A.5, A.6(b) and A.7 with $r_\Xi = d_\Xi$ and $r_K = d$. (2) for each n, the map $M_n :\ \Xi \times K \to K$ is weakly sequentially continuous. (3) the map $\overline{M} : K \to K$ is weakly sequentially continuous. (4) $\hat{\theta}_n \in K$ for all n $a.s. - I\!P$. Then all conclusions of Theorem 3.1 hold with $r_K = d$.

Remark 3.3:

(a) In a finite-dimensional Euclidean space, all topologies generated by the different metrics are equivalent, so that Corollary 3.2 includes the parametric results of KC (1978, p.73, theorem 2.5.1) as a special case. In particular, versions of the Ljung (1977), Marcet & Sargent (1989), Woodford (1990), and KW (1994) parametric adaptive learning results are included in Corollary 3.2.

(b) Any continuous linear functionals of $\hat{\theta}_n$ will converge to the same continuous linear functionals of any element of Θ^*. Moveover, a convex functional is the supremum of continuous linear functionals. Consequently, in a utility space or a profit space, value functions evaluated at $\hat{\theta}_n$ will converge almost surely to the values that would be achieved if learning were not necessary.

(c) In many cases, $\overline{M}$ is a gradient operator of a real-valued function $F \in C^1(\,I\!B\,, I\!R\,)$, which in turn is an objective function for some estimation problem. Now $\overline{M}(\theta) = 0$ for all $\theta \in \Theta^*$ gives all the critical points of F. If $\overline{M}$ is completely continuous, then F is continuous with respect to weak convergence in $\theta \in I\!B$ (see Berger, p.96 (ii)). In this case $\hat{\theta}_n \to \theta$ weakly implies $F(\hat{\theta}_n) \to F(\theta)$ strongly.

(d) If $\hat{\theta}_n \to \theta$ in weak topology in a Banach space, then there is a sequence $\{\,\overline{\theta}_n\,\}$, with $\overline{\theta}_n$ a convex combination of $\{\,\hat{\theta}_j :\ 1 \le j \le n\,\}$, converging to θ in norm (see, e.g., Berger, p.31 (1.3.11,iv)).

(e) If $\hat{\theta}_n \to \theta$ in weak topology in a uniformly convex Banach space (e.g., Hilbert spaces, L_p –spaces of integrable functions, Sobolev spaces of functions with generalized L_p derivatives, $1 < p < \infty$) with norm $\|\cdot\|$, and $\|\hat{\theta}_n\| \to \|\theta\|$, then $\hat{\theta}_n \to \theta$ in norm (see, e.g., Berger, p.31 (1.3.11,vi)).

(f) If Θ^* has only finitely many isolated elements, the particular element to which $\{ \hat{\theta}_n \}$ will converge depends on the initial conditions and the realization of the noise processes. Therefore, we cannot rule out the possibility of sunspot equilibria. The same things also occur in a correctly specified parametric learning model, e.g. Woodford (1990).

4. MODIFIED ALGORITHMS AND CONVERGENCE IN A HILBERT SPACE

In this section we specialize the above Banach-valued RMF and Corollary 3.2 to a real separable Hilbert space $\mathbb{H}$, with $K \subset \mathbb{H}$, and Ξ a (norm) bounded subset of a real separable Hilbert space. Let $\mathbb{H}$ be endowed with inner product $< \cdot , \cdot >$, norm $\| x \| = < x , x >^{1/2}$, and identity operator I. Again d and d_Ξ are the metrics corresponding to the weak topologies on $\mathbb{H}$ and Ξ respectively. For learning applications, we consider modifications of the above RMF procedure involving finite dimensional approximations, which is similiar to a Galerkin approximation to the Hilbert-valued RMF. The separability of $\mathbb{H}$ implies the existence of a complete orthonormal basis $\{ e_j ; j = 0,1,2,... \}$. Let $P_{k(n)} : \mathbb{H} \to \mathbb{H}_{k(n)}$ be the orthogonal projection of $\mathbb{H}$ onto $\mathbb{H}_{k(n)}$, where $\mathbb{H}_{k(n)} \equiv cl\, [sp\, (e_0 , e_1 , . . . , e_{k(n)})]$, and $\{ k(n) ; n = 0,1,2,...\}$ is an integer valued sequence such that:

$$k(0) = 0 , \quad k(n) \le k(n+1) \le k(n)+1 ; \quad \text{and} \qquad \lim_{n\to\infty} k(n) = \infty .$$

>From this it follows that $\dim (\mathbb{H}_{k(n)}) = k(n) + 1 \le n + 1$.

We replace Assumptions A.2 and A.5 with the following

ASSUMPTION B.2: Let Ξ be a norm-bounded subset of a separable Hilbert space with norm $\| \cdot \|_\Xi$ and d_Ξ the metric induced by the weak topology, and let $\mathbb{H}$ be a real separable Hilbert space with norm $\| \cdot \|$ and d the metric induced by the weak topology. For each n, the map $M_n : \Xi \times \mathbb{H} \to \mathbb{H}$ is (a) uniformly continuous on any bounded set of the form $\Xi \times \{ \theta \in \mathbb{H} : \| \theta \| \le B \}$ for any $0 < B < \infty$; and (b) weakly sequentially continuous.

ASSUMPTION B.5: $\hat{\xi}_0 : \Omega \to \Xi$ and $\hat{\theta}_0 : \Omega \to \mathbb{H}_{k(0)}$ are arbitrary measurable mappings independent of $\{ Z_n \}$, with $\|\hat{\theta}_0\| \le B < \infty$ $a.s.-I\!P$.

We can now define an RMF procedure with orthonormal projection (RMFP) in a Hilbert space as

$$\hat{\theta}_{n+1} = \hat{\theta}_n + a_n P_{k(n)} M_n(\hat{\xi}_n , \hat{\theta}_n) , \;\; \hat{\xi}_{n+1} = R_n(\hat{\xi}^n , \hat{\theta}^{n+1} , Z_{n+1}) , \quad n = 0,1,2,... .$$

In writing the above, we have abused notation by using $\hat{\theta}_n$ as the second argument of M_n and $\hat{\theta}^{n+1}$ as the second argument of R_n. Strictly speaking, these arguments should belong to $I\!H$ and $I\!H^{n+1}$; we could, for example, adhere to the formalities by replacing $\hat{\theta}_n$ with $\tilde{\theta}_n$, an element of $I\!H$ constructed by concatenating $\hat{\theta}_n$ with $0 \in I\!H$. We avoid doing this because the resulting notation becomes too heavy, and no confusion should result from our abuse.

Assumptions A.3, A.4 and A.7 continue to apply with (Ξ , r_Ξ) and (K , r_K) taken to be (Ξ , d_Ξ) and (K_B , d), where $K_B \equiv \{ \theta \in I\!H : \|\theta\| \le B \}$ for an arbitrary given $0 < B < \infty$.

ASSUMPTION B.6: There is a function $\overline{M} : I\!H \to I\!H$ such that (a) $\overline{M}$ is uniformly continuous on any norm-bounded subset of $I\!H$, (b) weakly sequentially continuous, and (c) for each $0 < T < \infty$, each $\theta \in I\!H$, and for any $\varepsilon > 0$

$$\lim_n I\!P \,[\, \sup_{j \ge n} \max_{t \le T} d\,(\Sigma_{m(jT) \le i \le m(jT+t)-1}\, a_i P_{k(i)} [\, M_i(\xi_i(\theta), \theta) - \overline{M}(\theta) \,] , 0) \ge \varepsilon \,] = 0 \; .$$

(Notice that most commonly used linear and nonlinear operators satisfy Assumptions B.2 and B.6(a)&(b). For example, bounded linear operators, Hölder and Lipschitz operators, uniformly continuous operators, some compact operators, some continuous operators with growth rate equal to or faster than linear, etc.)

THEOREM 4.1: Given A.1, B.2, A.3, A.4, B.5, B.6, and A.7 for the Hilbert-valued RMFP, if $\sup_n \|\hat{\theta}_n\| < \infty$ $a.s.-I\!P$, then:

(i) $\{ \theta^n(\cdot) \}$ is uniformly bounded and equicontinuous on bounded time intervals almost surely under the $\|\cdot\|-$topology.

(ii) $\{ \theta^n(\cdot) \}$ is sequentially compact under the weak topology; and any weak limit $\theta(t)$ satisfies the ODE $\dot{\theta} = \overline{M}(\theta(\cdot))$.

Let Θ^* be the set of locally asymptotically stable (in the sense of Liapunov) equilibria in $I\!H$ for the above ODE with domain of attraction $da(\Theta^*) \subset I\!H$. Fix a subset $C \subset da(\Theta^*)$, compact under the weak topology.

(iii) If $\hat{\theta}_n \in C$ *infinitely often*, then $\hat{\theta}_n \to \Theta^*$ as $n \to \infty$ $a.s. - I\!P$ in the weak topology, i.e., there exists Ω^* with $I\!P(\Omega^*) = 1$ such that for any $\omega \in \Omega^*$,

$$\lim_n [\, \inf_{\theta \in \Theta^*} |\, < \hat{\theta}_n(\omega), h > - < \theta, h > |\,] = 0, \quad \text{for all } h \in I\!H .$$

Remark 4.2: If we let K and Ξ be norm-bounded closed balls of real separable Hilbert spaces in Corollary 3.2, and let $P_{k(n)} = I$ for all n in Theorem 4.1, then they become akin to each other. Nevertheless, Theorem 4.1 has the extra assumptions B.2(a) and B.6(a), since K need not be a fixed ball in Theorem 4.1. Hence Theorem 4.1 is more applicable to models with no prior about where Θ^* belongs.

In the above theorems, we have assumed the uniform boundedness of $\{\hat{\theta}_n\}$ and $\{\hat{\xi}_n\}$. Because R_n embodies agent actions, it is plausible to assume that agents always choose their actions $\hat{\xi}_n$ from a bounded set Ξ. To establish the uniform boundedness of $\{\hat{\theta}_n\}$ generated by the RMFP procedure, one can adapt a traditional "fixed resetting" method from the parametric of RMF algorithms. This involves resetting $\hat{\theta}_n$ back into a fixed bounded set which contains all asymptotic equilibrium points whenever $\hat{\theta}_n$ attempts to escape. This method was proposed by Ljung (1977), and has been utilized by Marcet & Sargent (1989), Woodford (1990), KW (1994) and others. Here we consider a rather different resetting procedure, suggested by work of Yin and Zhu (1992). First we strengthen our condition on M_n by replacing B.2 with

ASSUMPTION C.2: (a) B.2(a) holds; (b) M_n is completely continuous on $\Xi \times I\!H$, $n = 0,1,2,\ldots$.
This accomodates the possibility that $\hat{\theta}_n$ may become unbounded.

Next, let $\{B_n;\ n = 0,1,2,\ldots\}$ be a sequence of strictly increasing positive real numbers with $\lim_{n\to\infty} B_n = \infty$. Define a sequence of positive integer-valued random variables by

$$T(0) = 0, \quad T(n+1) = T(n) + 1(J_n^{\,c}),$$

where $1(A)$ denotes the indicator of the set $A \in F$ and $J_n \equiv \{\, \| \hat{\theta}_n + a_n P_{k(n)} M_n(\hat{\xi}_n, \hat{\theta}_n) \| \le B_{T(n)} \}$.

A randomly truncated RMFP (TRMFP) in a Hilbert space is

$$\hat{\theta}_{n+1} = [\, \hat{\theta}_n + a_n P_{k(n)} M_n(\hat{\xi}_n, \hat{\theta}_n) \,]\, 1(J_n) + P_{k(n)} \bar{\theta}\, 1(J_n^{\,c}) \,,$$

$$\hat{\xi}_{n+1} = R_n(\hat{\xi}^{\,n}, \hat{\theta}^{\,n+1}, Z_{n+1}) \,, \quad n = 0,1,2,...,$$

where $\hat{\xi}_0$ and $\hat{\theta}_0$ are as in Assumption B.5 and $J_n^{\,c} \equiv \{ \| \hat{\theta}_n + a_n P_{k(n)} M_n(\hat{\xi}_n, \hat{\theta}_n) \| > B_{T(n)} \}$. We fix $0 < B < B_1$ and choose $\bar{\theta} \in I\!H$ to be an arbitrary fixed element with $\| \bar{\theta} \| < B$.

One problem with both the fixed resetting method and our TRMFP is the possibility of an infinite number of resettings, which will affect the limiting dynamics. To make sure that the resetting happens only a finite number of times almost surely, we strengthen Assumption B.6.

ASSUMPTION C.6: There is a function $\overline{M} : I\!H \to I\!H$ such that (a) B.6(a) holds; (b) B.6(b) holds; (c) for any $0 < T < \infty$, for any $\varepsilon > 0$ and $\| \theta \| \le \bar{B}$,

$$\lim_n I\!P \; [\, \sup_{j \ge n} \max_{t \le T} \| \Sigma_{m(jT) \le i \le m(jT+t)-1} \, a_i \, P_{k(i)} \, [\, M_i(\xi_i(\theta), \theta) - \overline{M}(\theta) \,] \| \ge \varepsilon \,] = 0 \;.$$

We also add

ASSUMPTION C.8: $\Theta^* \equiv \{ \theta \in I\!H : \overline{M}(\theta) = 0 \}$ is a compact subset of $I\!H$.

(Hence $\sup_{\theta \in \Theta^*} \| \theta \| < \infty$.)

ASSUMPTION C.9: There is a bounded and twice continuously Frechet differentiable functional $V : I\!H \to I\!R$ such that: (a) $V(\theta) \ge 0$ for $\theta \in I\!H$, $\lim_{\|\theta\| \to \infty} V(\theta) = \infty$.

(b) $V(\theta) \ne V(\theta^*)$, for $\theta \in I\!H - \Theta^*$ and $\theta^* \in \Theta^*$.

(c) $< V'(\theta), \overline{M}(\theta) > \, < 0$, for all $\theta \in I\!H - \Theta^*$.

(d) $V(\bar{\theta}) < b$ and $[\, V(\bar{\theta}), b \,] \cap V(\Theta^*) \ne [\, V(\bar{\theta}), b \,]$, where $b \equiv \inf \{ V(\theta) : \| \theta \| = B \}$.

THEOREM 4.3: Given the Hilbert-valued TRMFP, suppose A.1, C.2, A.3, A.4, B.5, C.6, A.7, C.8 and C.9 hold. Then $\hat{\theta}_n \to \Theta^*$ as $n \to \infty$ $a.s.-I\!P$ in the weak topology.

Instead of strengthening Assumption B.2 to C.2, we can strengthen Assumption A.7 to C.7

ASSUMPTION C.7: A.7 holds with $r_K = d$, but $r_\Xi = \| \cdot \|_\Xi$.

THEOREM 4.4: Given the Hilbert-valued TRMFP, suppose A.1, B.2, A.3, A.4, B.5, C.6, C.7, C.8 and C.9 hold. Then $\hat{\theta}_n \to \Theta^*$ as $n \to \infty$ $a.s.-I\!P$ in the weak topology.

Remark 4.5: Theorems 4.1, 4.3 and 4.4 still hold when $P_k \equiv I$ for all k, that is, in the case of no orthogonal projections, hence they imply all the conclusions of Corollary 3.2.

5. SUFFICIENT CONDITIONS FOR ALMOST-SURE CONVERGENCE

Although Assumptions B.6(c) and A.7 (C.6(c) and C.7) seem to be rather abstract, they are very mild conditions. Assumption B.6 (C.6) is satisfied for many Hilbert space-valued dependent processes, and Assumption A.7 (C.7) is satisfied by imposing appropriate continuity conditions on the mappings $\{ R_n \}$. We provide some sufficient conditions for B.6(c) and A.7 (C.6(c) and C.7) in this section. Using Theorems 4.3 or 4.4, we can assume $\sup_n \| \hat{\theta}_n \| \le \bar{B} < \infty$ $a.s.-I\!P$ without loss of generality.

Let $\theta^{n+1} \equiv (\theta, \cdots, \theta)$, the point in $I\!H^{n+1}$ with identical coordinate θ in each position, and define $\xi_n(\theta)$ recursively as

$$\xi_{n+1}(\theta) = R_n(\xi^n(\theta), \theta^{n+1}, Z_{n+1}), \quad n = 0,1,2,\ldots; \quad \xi_0(\theta) = \xi^0(\theta) = \hat{\xi}_0 .$$

If $\bar{M}(\theta) = \lim_{n\to\infty} E[M_n(\xi_n(\theta), \theta)]$ exists in the norm topology (i.e., $\lim_{n\to\infty} \| E[M_n(\xi_n(\theta), \theta)] - \bar{M}(\theta) \| = 0$), then it will suffice for Assumption C.6(c) (hence B.6(c)) that for each θ such that $\| \theta \| \le \bar{B}$,

$$\| \sum_{j=0}^{\infty} a_j [P_{k(j)} M_j(\xi_j(\theta), \theta) - E(P_{k(j)} M_j(\xi_j(\theta), \theta))] \| < \infty \quad a.s.-I\!P .$$

This summability condition can be verified using convergence results for Hilbert space-valued mixingales recently obtained by Chen and White (CW) (1994). For convenience, we now provide the essential background and definitions. For further detail, see CW (1994).

Let $\| \cdot \|_p$ denote the L_p–norm for a Hilbert-valued random element ($I\!H-r.e.$) X, $\| X \|_p \equiv [E \| X \|^p]^{1/p}$, $1 \le p < \infty$. McLeish (1975) first introduced the notion of an L_2–mixingale sequence for real random variables; the notion of L_p–mixingales was introduced by Andrews (1988). Gyorfi and Masry (1989) present a notion of an L_2–mixingale for Hilbert space-valued random elements. Here we use the notion of an $I\!H$–valued L_p –mixingale process (CW, 1994) :

DEFINITION 5.1: Let $\{ W_n ; -\infty < n < \infty \}$ be a sequence of $I\!H$– *r.e.´s* with finite L_p –norms, $1 \le p < \infty$. Let $\{ A^n \}$ be a filtration of F . Then $\{ W_n , A^n \}$ is an $L_p(I\!H)$ *–mixingale* sequence if there exist sequences of finite nonnegative constants $\{ c_n ; n \ge 1 \}$ and $\{ \psi_m ; m \ge 0 \}$ with $\psi_m \to 0$ as $m \to \infty$ such that the following two inequalities hold for all $n \ge 1$, $m \ge 0$:

$$\| E (W_n \mid A^{n-m}) \|_p \le \psi_m c_n ;$$

$$\| W_n - E (W_n \mid A^{n+m}) \|_p \le \psi_{m+1} c_n .$$

If, in addition, W_n is A^n –measurable, then $\{ W_n , A^n \}$ is an *adapted* $L_p(I\!H)$ –mixingale sequence.

An $L_p(I\!H)$ –mixingale with $1 \le p < \infty$ has zero mean. We can choose $\{ \psi_m ; m \ge 0 \}$ to be non-increasing in m when $\{ W_n , A^n \}$ is an adapted L_p– mixingale ($p \ge 1$). We say that ψ_m is of size–a if $\sum_{m=0}^{\infty} [\psi_m]^\delta < \infty$ or $\psi_m = o (m^{-1/\delta})$ for some $a < (1/\delta)$ or $\psi_m = O (m^\lambda)$ for some $\lambda < -a$.

To exploit CW’s (1994) convergence results, we replace Assumption C.6 as follows

ASSUMPTION D.6: (a) For each $\theta \in I\!H$ such that $\|\theta\| \le \bar{B}$, $\bar{M}(\theta) = \lim_{n\to\infty} E [M_n(\xi_n(\theta) , \theta)]$ exists in the norm topology; (b) the map $\theta \to \bar{M}(\theta)$ is uniformly continuous on $\{ \theta \in I\!H : \|\theta\| \le \bar{B} < \infty \}$, and weakly sequentially continuous; (c) for each $\|\theta\| \le \bar{B}$,

$$\{ P_{k(n)} M_n(\xi_n(\theta) , \theta) - E [P_{k(n)} M_n(\xi_n(\theta) , \theta)] , F^n ; \ n = 0,1,2,... \}$$

is an $L_2(I\!H)$ –mixingale with parameters $\{c_n\}$ and $\{\psi_m\}$ satisfying $\sum_{n=0}^{\infty} (c_n a_n)^2 < \infty$ $a.s. - I\!P$, and $\psi_m = O (m^{-1/2} (\log m)^{-2})$, where $\sigma(Z_0 , . . . , Z_n) \subset F^n$.

Note that when $P_{k(n)} M_n(\xi_n(\theta) , \theta)$ is norm bounded uniformly in n , we have $\sup_n c_n \le c < \infty$, and the requirement of $\sum_{n=0}^{\infty} (c_n a_n)^2 < \infty$ $a.s. - I\!P$ becomes the commonly seen condition $\sum_{n=0}^{\infty} a_n^2 < \infty$ $a.s. - I\!P$ in parametric models. Although the boundedness uniformly in n is a reasonable assumption in parametric setting, it often fails to hold in nonparametric models; see the examples in Section 6.

The following result is a simple application of a strong law of large numbers for Hilbert space-valued mixingale processes (corollary 3.8 in CW, 1994), and the triangle inequality.

LEMMA 5.2: Assumptions A.1, A.3, A.4 and D.6 imply Assumption C.6 (hence B.6).

The following assumptions imply C.7 (hence A.7)

ASSUMPTION D.1(a): Assumption A.1 holds when G is a norm-bounded convex subset of a real separable Hilbert space with norm $\|\cdot\|_G$, containing the support of Z_n, $n = 0,1,2,\ldots$.

ASSUMPTION D.4(a)&(b): Let Ξ be a norm-bounded subset of a separable Hilbert space, and $\mathbb{H}$ a separable Hilbert space. Let $\rho_n : \Xi \times \mathbb{H} \times G \to \Xi$ be a mapping bounded uniformly in n, and continuous such that:

(a) for each $(\theta, z) \in \mathbb{H} \times G$, the mapping $\rho_n(\cdot, \theta, z) : (\Xi, \|\cdot\|_\Xi) \to (\Xi, \|\cdot\|_\Xi)$ is contractive uniformly in (θ, z, n); i.e., there exists a c_0 independent of (θ, z, n) with $0 \le c_0 < 1$, such that for any $\xi_1, \xi_2 \in \Xi$,

$$\|\rho_n(\xi_1, \theta, z) - \rho_n(\xi_2, \theta, z)\|_\Xi \le c_0 \|\xi_1 - \xi_2\|_\Xi .$$

(b) for each $(\xi, z) \in \Xi \times G$, the mapping $\rho_n(\xi, \cdot, z) : (\mathbb{H}, d) \to (\Xi, \|\cdot\|_\Xi)$ is Lipschitz continuous on $\{\theta : \|\theta\| \le \bar{B} < \infty\}$, uniformly in (ξ, z, n); i.e., there exists a c_1 independent of (ξ, z, n) with $0 \le c_1 < \infty$, such that for any $\theta_1, \theta_2 \in \mathbb{H}$, $\|\theta_1\| \le \bar{B}$, $\|\theta_2\| \le \bar{B}$,

$$\|\rho_n(\xi, \theta_1, z) - \rho_n(\xi, \theta_2, z)\|_\Xi \le c_1\, d(\theta_1, \theta_2) .$$

LEMMA 5.3: Recursively define $\xi_{n+1}(\{\theta_i\}) = \rho_n(\xi_n(\{\theta_i\}), \theta_{n+1}, Z_{n+1})$ for any sequence $\{\theta_i\}$ with $\theta_i \in \mathbb{H}$, and all $i, n = 0,1,2,\ldots$. Then Assumptions D.1(a), D.4(a)&(b) imply Assumption C.7 (hence A.7).

Assumptions more primitive than D.6(c) can be given using notions of mixing and near epoch dependence.

DEFINITION 5.4: (1) Let A, G be two σ–subfields on the probability space (Ω, F, P). Define two measures of dependence as :

$$\alpha\,(\,A\,,\,G\,) \equiv \sup\,[\;|\;P(\,A\;C\,) - P(A)\,P(C)\;|\;:\;A \in A\,,\,C \in G\,]\,;$$

$$\phi\,(\,A\,,\,G\,) \equiv \sup\,[\;|\;P(\,C\;|\;A\,) - P(C)\;|\;:\;A \in A\,,\,P(A) > 0\,,\;C \in G\,]\,.$$

(2) Let $\{\,D_n\,\}$ be a sequence of Banach-valued random elements ($I\!B$ –*r.e.´s*) defined on the probability space $(\,\Omega\,,\,F\,,\,P\,)$, and denote $A_a{}^b \equiv \sigma\,(\,D_j\,;\;a \le j \le b\,)$. Define

$$\alpha(m) \equiv \sup_n\,[\,\alpha\,(\,A_{-\infty}{}^n\,,\,A_{n+m}{}^{\infty}\,)\,]\,;\quad \phi(m) \equiv \sup_n\,[\,\phi\,(\,A_{-\infty}{}^n\,,\,A_{n+m}{}^{\infty}\,)\,]\,.$$

If $\lim_{m\to\infty}\alpha(m) = 0$, then $\{\,D_n\,\}$ is called an α–*mixing* sequence. If $\lim_{m\to\infty}\phi(m) = 0$, then $\{\,D_n\,\}$ is called a ϕ–*mixing* sequence.

(3) Let $\{\,D_n\,;\;-\infty < n < \infty\,\}$ be a $I\!B$–*r.e.* sequence and $\{\,W_n\,;\;-\infty < n < \infty\,\}$ be an $I\!H$–*r.e.* sequence. Then $\{\,W_n\,\}$ is called $L_p(I\!H)$–*near epoch dependent* (NED) on $\{\,D_n\,\}$ if $\|\,W_n\,\|_p < \infty$, $1 \le p < \infty$, and there exist constants $\{\,\mu_m \ge 0\,;\;m \ge 0\,\}$ with μ_m decreasing to zero as $m \to \infty$ and $\{\,d_n \ge 0\,;\;n \ge 1\,\}$ with $\sup_n d_n < \infty$ such that

$$\|\,W_n - E\,[\,W_n\;|\;A_{n-m}{}^{n+m}\,]\,\|_p \le \mu_m\,d_n\,,\quad \text{where } A_a{}^b \text{ is as in (2)}\,.$$

CW (1994, lemma 4.3) have shown that zero mean $L_p(I\!H)$–NED processes are $L_p(I\!H)$–mixingales. We now show how to exploit this fact to give more primitive assumptions on the underlying random noise $\{\,Z_n\,\}$ to generate an $L_2(I\!H)$–NED process which satisfies D.6(c).

ASSUMPTION D.1(b): $\{\,Z_n\,;\,n = 0,1,2,...\,\}$ has finite L_r–norm with $r \ge 2$, and is a sequence of $L_2(G)$–NED functions on $\{\,D_n\,\}$ of size $-1/2$, where $\{\,D_n\,\}$ is a $I\!B$-valued mixing process on $(\,\Omega\,,\,F\,,\,I\!P\,)$ with ϕ_m of size $-1/2$ or α_m of size -1 .

ASSUMPTION D.4(c): The mapping $\rho_n(\,\xi\,,\theta\,,\,\cdot\,) : (\,G\,,\|\cdot\|_G\,) \to (\,\Xi\,,\|\cdot\|_\Xi\,)$ is Lipschitz continuous uniformly in $\xi \in \Xi\,,\theta \in I\!H$ and $n \ge 0$; i.e., there exists a c_2 independent of $(\,\xi\,,\theta\,,\,n\,)$ with $0 \le c_2 < \infty$, such that for any $z_1\,,\,z_2 \in G$,

$$\|\rho_n(\,\xi\,,\theta\,,\,z_1\,) - \rho_n(\,\xi\,,\theta\,,\,z_2\,)\|_\Xi \le c_2\,\|\,z_1 - z_2\,\|_G\,.$$

ASSUMPTION D.2: (a) B.2 holds, and (b) for each n , there exists a $c_{3,n}$, independent of

$\theta \in I\!H$, $\|\theta\| \le \bar{B}$,

with $0 \le c_{3,n} < \infty$, such that for any $\xi_1 , \xi_2 \in \Xi$,

$$\| P_{k(n)} M_n(\xi_1 , \theta) - P_{k(n)} M_n(\xi_2 , \theta) \| \le c_{3,n} \|\xi_1 - \xi_2\|_\Xi .$$

ASSUMPTION D.3: A.3 holds, and $\sum_{n=0}^{\infty} (c_{3,n} a_n)^2 < \infty$ $a.s. - I\!P$.

LEMMA 5.5: Recursively define $\xi_{n+1}(\theta) = \rho_n(\xi_n(\theta) , \theta , Z_{n+1})$ for any constant sequence $\{\theta\}$ with $\theta \in I\!H$ and all $n = 0,1,2,...$. Then Assumptions D.1, D.2(b), D.3, D.4(a)&(c) imply Assumption D.6(c).

COROLLARY 5.6: Suppose that Assumptions D.1 - D.4, B.5, D.6(a)&(b), C.8 and C.9 hold for the Hilbert -valued TRMFP, with $\hat{\xi}_{n+1} = \rho_n(\hat{\xi}_n , \hat{\theta}_{n+1} , Z_{n+1})$, $n = 0,1,2,...$. Then all conclusions of Theorem 4.1 hold.

6. APPLICATIONS

We are now in a position to obtain convergence results for versions of all the examples in section 2 by applying the foregoing theory.

In Example 2.1 using Method 1 we denote $\xi_t' \equiv (\xi_{1,t} , \xi_{2,t})$ and $Z_t' \equiv (Z_{1,t} , Z_{2,t})$, where $Z_{1,t} \equiv X_t$ and $Z_{2,t} \equiv u_t$. Let $\xi_{1,t} = Z_{1,t}$ and $\xi_{2,t}(\theta) = \hat{p}_t = a(Z_{1,t}) + b\, \theta(Z_{1,t}) + Z_{2,t}$, with

$$M_t(\xi_t , \theta(\cdot)) = (h_t)^{-1} K((\cdot - \xi_{1,t}) / h_t) \times [\xi_{2,t} - \theta(\cdot)] .$$

Also $\Xi \subset I\!R^2$ (to be specified later), $I\!H = L_2([-1,1])$, and

$$\|\xi\|_\Xi^2 = | \xi_1 |^2 + | \xi_2 |^2 \ ; \quad \|\theta\|^2 = \int_{[-1,1]} [\theta(x)]^2 dx \ .$$

Then we get a Hilbert ($I\!H$) -valued RMF: $\hat{\theta}_{t+1} = \hat{\theta}_t + (t+1)^{-1} M_t(\hat{\xi}_t , \hat{\theta}_t)$. We obtain the following result by applying Corollary 5.6 and Remark 4.5.

PROPOSITION 6.1: In Example 2.1 and Method 1, suppose $\{ u_t \}$ is *i.i.d.* sequence with compact support $[l_1,l_2]$, zero mean, finite variance, and independent of $\{ X_t \}$. Suppose further that the following hold:

(6.1.1) each element of $\{ X_t \, ; t \geq 0 \}$ has compact support $[-1,1]$ and the same unknown continuous density f ;

(6.1.2) $\{X_t\}$ is $L_2([-1,1])$ –NED with $\sup_t d_t < \infty$ and size $-1/2$ on a mixing sequence $\{D_t\}$ with ϕ of size $-1/2$ or α of size -1 ;

(6.1.3) $K(\cdot)$ is Lipschitz continuous, $K(\cdot) \geq 0$, symmetric about zero, $\int_{\boldsymbol{R}} K(x)\, dx = 1$, and $\int_{\boldsymbol{R}} \mid K(x) \mid^2 dx \leq C < \infty$;

(6.1.4) $\{ h_t \}$ is a decreasing sequence of positive numbers satisfying $h_t = O\,(\,(t+1)^{-\delta}\,)$ for some fixed $0 < \delta < 1/4$;

(6.1.5) a is Lipschitz continuous on $[-1,1]$;

(6.1.6) $\hat{\theta}_0$ is an arbitrary random Lipschitz continuous function on $[-1,1]$, independent of $\{ X_t \}$ and $\{ u_t \}$.

Then $\mid \int_{\boldsymbol{R}} (\,\hat{\theta}_t(x) - a(x)\,(1-b)^{-1}\,)\, h(x)\, dx \mid \to 0$ as $t \to \infty$ $a.s. - I\!P$ for all $h(\cdot) \in L_2([-1,1])$. □

In Example 2.2, we denote $\hat{\theta}_t = \theta_t$, $\xi_t' = (\,\xi_{1,t}\,, \xi_{2,t}\,)$ with $\xi_{1,t} = x_t$, $\xi_{2,t} = Z_t$, and

$$M_t(\,\xi_t\,, \theta(\,\cdot\,)\,) = Q(\xi_t)\, K(\,(\,\cdot - \xi_{1,t}\,)\,/\,h_t\,)\,[h_t]^{-1} - \theta(\,\cdot\,) \int_{[0,1]} Q(\xi_t)\, K(\,(\,y - \xi_{1,t}\,)\,/\,h_t\,)\,[h_t]^{-1}\, dy \,.$$

Here $\Xi = [0,1] \times G$ and $I\!H = L_2([0,1])$, where G is a bounded subset of a separable Hilbert space.

$$\|\xi\|_{\Xi} = (\, \mid \xi_{1,t} \mid^{\,2} + \|\xi_{2,t}\|_G^{\,2}\,)^{\,1/2}\,; \quad \|\theta\| = (\,\int_{[0,1]} [\,\theta(y)\,]^2\, dy\,)^{\,1/2}\,.$$

PROPOSITION 6.2: In Example 2.2, $S_0(\cdot) : [0,1] \to (0,\infty)$ is chosen to be continuous in $x \in [0,1]$, and independent of $\{ Z_t \}$. Suppose for each $t \geq 0$, x_t is independent of $\{ Z_j \}$, and the following hold:

(6.2.1) $\{ Z_t \}$ is a bounded Hilbert -valued random sequence with bounded support G , and is $L_2(G)$ –NED with parameters $\{d_t\}$ and $\{\mu_m\}$ of size $-1/2$ on a mixing sequence $\{D_t\}$ with ϕ of size $-1/2$ or α of size -1 .

(6.2.2) $Q : [0,1] \times G \to (0,\infty)$ has the form $Q(x,z) \equiv Q_1(x) \times Q_2(z)$ for all $x \in [0,1]$, $z \in G$, where $Q_1(\cdot)$ and $Q_2(\cdot)$ are both Lipschitz continuous, and $Q_2(\,Z_t\,)$ has same finite mean for all t .

(6.2.3) $K(\cdot)$ is Lipschitz continuous, $K(\cdot) \geq 0$, symmetric about zero, $\int_{\boldsymbol{R}} K(x)\, dx = 1$, and $\int_{\boldsymbol{R}} \mid K(x) \mid^2 dx \leq C < \infty$;

(6.2.4) $h_t = O(\,(t+1)^{-\delta}\,)$ for some $\delta \in (0,1/4)$.

(6.2.5) Θ^* is compact and not empty, where

$$\Theta^* \equiv \{\theta : \theta(x) \lim E[\, Q(x,Z_t) - \int_{[0,1]} Q(s,Z_t)\,\theta(s)\, ds] = 0 \text{ for all } x \in [0,1]\, ;\ \theta \text{ is a continuous density on } [0,1]\,\}.$$

Then $\hat{\theta}_t \to \Theta^*$ in the weak topology almost surely.

(Hence, in the limit, $\hat{\theta}_t$ will put positive weight in a small area where the expected payoff is maximized and zero weight elsewhere). □

In Example 2.3 we let $Z_t = r_t$, $\theta_t = W_t^e$, $\xi_t = \hat{W}_t = f(\theta_t, Z_t)$; then $M_t = \xi_t - \theta_t$. The following result is a simple application of Corollary 5.6.

PROPOSITION 6.3: In Example 2.3, suppose

(6.3.1) $U(\cdot)$ is twice continuously differentiable with $U' > 0$ and $U'' < 0$.

(6.3.2) $\hat{\theta}_0$ is chosen from a twice continously differentiable family such that $\hat{\theta}_0' > 0$ and $\hat{\theta}_0'' < 0$ almost surely.

(6.3.3) $\{ r_t \}$ is a positive, i.i.d. random sequence with a compact support $[\, a\, , b\,]$.

Then $|\int_{\mathbb{R}} (\hat{\theta}_t(x) - \hat{\xi}_t(x))\, h(x)\, dx\,| \to 0$ as $t \to \infty$ $a.s. - I\!P$, and $|\int_{\mathbb{R}} (\hat{\xi}_t(x) - W(x))\, h(x)\, dx\,| \to 0$ as $t \to \infty$ $a.s. - I\!P$, for all $h(\cdot) \in L_2(I\!R)$, where $W(\cdot)$ is the true value function without learning. □

In Example 2.4 we define $\theta' \equiv (\theta_{12}, \theta_{21})$, $\xi_t' \equiv (\xi_{1,t}, \xi_{2,t})$, $M_t' \equiv (M_{1,t}, M_{2,t})$, and $Z_t' \equiv (Z_{1,t}, Z_{2,t})$; then $\xi_{t+1} = \int_{[0,1]} x\, \theta(x)\, dx - Z_{t+1}$ for the quadratic payoff function, and

$$M_t'(\xi_t, \theta_t(\cdot)) = (\, [h_{1,t}]^{-1} K_1((\,\cdot - (0,1)\xi_t)/h_{1,t}),\, [h_{2,t}]^{-1} K_2((\,\cdot - (1,0)\xi_t)/h_{2,t})) - \theta_t(\cdot)'\, .$$

Also $G \subset I\!R^2$ (to be specified later), $\Xi = [0,1] \times [0,1] \in I\!R^2$, $I\!H = L_2([0,1]) \times L_2([0,1])$. For any $\xi' = (\xi_1, \xi_2) \in \Xi$ and any $\theta' = (\theta_1, \theta_2) \in I\!H$, put

$$\|\xi\|_{\Xi}^2 \equiv |\,\xi_1\,|^2 + |\,\xi_2\,|^2\, ; \quad \|\theta\|^2 \equiv \int_{[0,1]} [\,\theta_1(x)\,]^2\, dx + \int_{[0,1]} [\,\theta_2(x)\,]^2\, dx\ .$$

PROPOSITION 6.4: In Example 2.4 with quadratic payoff function, let $\hat{\xi}_{1,0}$, $\hat{\xi}_{2,0}$ be arbitrary $(0,1)$ –valued random variables, and let $\hat{\theta}_{12,0}$, $\hat{\theta}_{21,0}$ be arbitrary $L_2([0,1])$ –valued random continuous density functions. Suppose that $\hat{\xi}_{1,0}, \hat{\xi}_{2,0}$, $\hat{\theta}_{12,0}$, $\hat{\theta}_{21,0}$, $\{Z_{1,t}\}$ and $\{Z_{2,t}\}$ are mutually independent.

Further suppose for $i = 1 , 2$, that the following hold:

(6.4.1) For all t , the elements of $\{\xi_{i,t}\}$ have compact support $[0,1]$ and unknown continuous density function f_i .

(6.4.2) $\{Z_{i,t}\}$ is uniformly bounded in t , and is L_2 –NED of size $-1/2$ on a mixing sequence $\{D_t\}$ with ϕ of size $-1/2$ or α of size -1 .

(6.4.3) $K_i(\cdot)$ is Lipschitz continuous, symmetric about zero, $K_i(\cdot) \geq 0$, $\int_{\mathbf{R}} K_i(x)\, dx = 1$, and $\int_{\mathbf{R}} | K_i(x) |^2 \, dx \leq C_i < \infty$.

(6.4.4) $\{ h_{i,t} \}$ is a decreasing sequence of positive numbers satisfying $h_{i,t} = O\left((t+1)^{-\delta_i} \right)$ for some fixed $0 < \delta_i < 1/4$.

Then $$\left| \int_{[0,1]} \left(\hat{\theta}_{12,t}(x) - f_2(x) \right) h(x)\, dx \right| + \left| \int_{[0,1]} \left(\hat{\theta}_{21,t}(y) - f_1(y) \right) g(y)\, dy \right| \to 0 \quad \text{as } t \to \infty \;\; a.s. - I\!P ,$$ for all $h(\cdot) , g(\cdot) \in L_2([0,1])$. □

7. SUMMARY

The algorithms introduced in this paper provide tools for economists to study adaptive learning models for economic agents, and for econometricians to estimate nonparametric time series models with dynamic latent variables. The procedures are easy to compute and allow time-dependent noise and non-linear updating functions. In particular, both economic agents and econometricians can learn the "truth" almost surely under favorable conditions using these procedures, regardless of their priors. We can thus avoid the pitfalls possible using parametric approaches. For example, in Marcet - Sargent type parametric learning models, if agents use correctly specified parametric models to learn, they will learn the REE almost surely; but if agents use incorrect parametric models to learn, they can easily arrive at incorrect belief equilibria.

An important area for future research concerns the rate of convergence of the estimates $\hat{\theta}_n$. This has important implications for the optimal growth rate of $k(n)$. We expect that our nonparametric estimator will converge more slowly than their parametric conterparts; we also plan to compare the rate of convergence of our procedures to those of other nonparametric estimation procedures.

Another important area for further reseach is the investigation of asymptotic distribution properties. This represents a significant challenge, because as far as we know, there are no such results even for parametric procedures with feedback.

8. MATHEMATICAL APPENDIX

Definitions and Notations:

Let X and Y be Banach spaces with norms $\|\cdot\|_X$ and $\|\cdot\|_Y$ respectively. Let $M : X \to Y$ be an operator. Then M is (*norm*) *continuous* provided $x_n \to x$ in norm–$\|\cdot\|_X$ implies $M(x_n) \to M(x)$ in norm–$\|\cdot\|_Y$. M is *completely continuous* (or *strongly continuous*) provided $x_n \to x$ in the weak topology of X implies $M(x_n) \to M(x)$ in norm–$\|\cdot\|_Y$. M is *weakly sequentially continuous* (or *demicontinuous*) provided $x_n \to x$ in the weak topology of X implies $M(x_n) \to M(x)$ in the weak topology of Y. M is *uniformly continuous* on $D \subseteq X$ provided for any $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$ such that for any $x, u \in D$ with $\|x-u\|_X < \delta(\varepsilon)$, we have $\|M(x)-M(u)\|_Y < \varepsilon$. M is *Lipschitz continuous* on $D \subseteq X$ provided there exists a fixed c such that for any $x, u \in D$, we have $\|M(x)-M(u)\|_Y \le c\,\|x-u\|_X$. M is *contractive* provided M is Lipschitz continuous with $0 \le c < 1$. M is *bounded* provided M maps bounded sets into bounded sets. M is *compact* provided M is continuous and maps bounded sets into relatively compact sets. The relations among the above various operators can be found in in Berger (1977, pp 64-65, 89,93,etc).

PROOF OF THEOREM 3.1: (i) Let Ω' denote the union of the set where $\{\hat{\theta}_n\}$ is not in K with the union of the exceptional sets in A.3, A.6(b) and A.7. Fix $\omega \in \Omega - \Omega'$. Given $\hat{\theta}_n \in K$ for all n and that K is convex, we have $\theta^n(t) \in K$ for each t and all n. To prove equicontinuity of $\{\theta^n(\cdot)\}$, it suffices to show that for each $s > 0$, $r_K(\theta^n(t+s), \theta^n(t)) \le \varepsilon(s)$ for all n large enough, where $\varepsilon(s) \to 0$ as $s \to 0$. Denote $i_0 \equiv m(t_n+t) = \max\{\, l \ge 0 : \sum_{j=n}^{l-1} a_j \le t \,\}$ and $i_s \equiv m(t_n+t+s) - 1$. Since $\sum_{j=0}^{\infty} a_j = \infty$ ((A.3)), for any finite $s > 0$, $t > 0$, we have $\limsup_{n\to\infty} [\, i_s - i_0 \,] < \infty$. By the definition of θ^n, we have

$$\theta^n(t+s) - \theta^n(t) = \Sigma_{i_0+1 \le j \le i_s-1}\, a_j\, M_j(\hat{\xi}_j, \hat{\theta}_j) + \alpha\, a_{i_0}\, M_{i_0}(\hat{\xi}_{i_0}, \hat{\theta}_{i_0}) + \beta\, a_{i_s}\, M_{i_s}(\hat{\xi}_{i_s}, \hat{\theta}_{i_s}),$$

for some $\alpha, \beta \in [0,1]$. Let $\delta \equiv \Sigma_{i_0+1 \le j \le i_s-1}\, a_j$, then $\delta \le s$. Hence $r_K(\theta^n(t+s), \theta^n(t)) \le J_1 + J_2 + J_3$, where

$$J_1 \equiv r_K\,(\,\delta\,\Sigma_{i_0+1\le j\le i_s-1}\,(\,a_j\,/\,\delta\,)\,M_j(\,\hat{\xi}_j\,,\hat{\theta}_j\,)\,,0\,)\,,$$

and

$$J_2 \equiv r_K\,(\,a_{i_0}\,\alpha\,M_{i_0}(\,\hat{\xi}_{i_0}\,,\hat{\theta}_{i_0}\,)\,,0\,)\,,\quad J_3 \equiv r_K\,(\,a_{i_s}\,\beta\,M_{i_s}(\,\hat{\xi}_{i_s}\,,\hat{\theta}_{i_s}\,)\,,0\,)\,.$$

Given A.2(b) (boundedness), A.3, and $\delta \le s \to 0$, by exercise 4(d) in Conway (1991, p. 107), we have $J_1 \to 0$, $J_2 \to 0$, and $J_3 \to 0$ as $s \to 0$ for all n large enough. Hence $\{\,\theta^n(\cdot)\,\}$ is equicontinuous under the r_K –metric.

(ii) Given (i) and $(\,K\,,r_K\,)$ a compact space, we can apply Ascoli - Arzela's lemma to conclude that $\{\,\theta^n(\cdot)\,\}$ is sequentially compact. Thus, every sequence of $\{\,\theta^n(\cdot)\,\}$ has a convergent subsequence in the r_K –metric. For notational simplicity, we denote the convergent subsequence again by $\{\,\theta^n(\cdot)\}$ and the limit (in r_K –metric) by $\theta(\cdot)$. Moreover, the convergence is uniform on each compact t – set. i.e., $\sup_{0\le t\le T} r_K\,(\,\theta^n(t)\,,\theta(t)\,) \to 0$ as $n \to \infty$ $a.s.-I\!P$ for any $0 \le T < \infty$. Now fix any $0 < T < \infty$ and let $t\,,\,t+s \in [\,0\,,T\,]$. For this convergent subsequence $\{\,n\,\}$, given A.2(b) and A.3, we have

$$(a.1)\quad \theta^n(t+s) - \theta^n(t) = \Sigma_{\,m(t_n+t)\le j\le m(t_n+t+s)-1}\;a_j\,M_j(\,\hat{\xi}_j\,,\hat{\theta}_j\,) - o(1)\,,$$

where $\sup_{0\le t,t+s\le T} r_K(\,o(1)\,,0\,) \to 0$ as $n \to \infty$. Given A.3, we can chose a sequence $\{\,\delta_n > 0\,\}$ such that

$$\delta_n \downarrow 0 \quad\text{and}\quad \delta_n^{-1}\sup_{j\ge n} a_j \to 0 \quad\text{as}\quad n \to \infty\,.$$

We select an increasing integer - valued sequence $\{\,r_l\,\}$ with $r_1 = n$ such that $\Sigma_{\,r_l\le j\le r_{l+1}-1}\,a_j = \delta_n$, modulo an end value. Hence $\delta_n^{-1}\,(\,t_{r_{l+1}} - t_{r_l}\,) \to 1$ as $n \to \infty$. Let $0 < \delta_n < 2\,\delta_n < \cdots \le t+s$ be a partition of $[\,0\,,t+s\,]$ with equal length of subinterval δ_n.

$$\Sigma_{\,m(t_n+t)\le j\le m(t_n+t+s)-1}\;a_j\,M_j(\,\hat{\xi}_j\,,\hat{\theta}_j\,)$$

$$= \Sigma_l\,\delta_n \times [\,\delta_n^{-1}\,\Sigma_{r_l\le j\le r_{l+1}-1}\;a_j\,M_j(\,\hat{\xi}_j\,,\hat{\theta}_j\,)\,]$$

$$= \Sigma_l\,\delta_n\,f_n(\tau) + \Sigma_l\,\delta_n\,A_n(\tau) = \int_{t\le\tau<t+s} f_n(\tau)\,d\tau + \int_{t\le\tau<t+s} A_n(\tau)\,d\tau\,,$$

where $f_n(\cdot)$ and $A_n(\cdot)$ are the piecewise right continuous constant interpolations for $\tau \in [\,t_{r_l} - t_n\,,\,t_{r_{l+1}} - t_n\,)$,

$$f_n(\tau) \equiv \delta_n^{-1} \Sigma_{r_l \le j \le r_{l+1}-1} \; a_j \, \overline{M}(\hat{\theta}_j)$$

and

$$A_n(\tau) \equiv \delta_n^{-1} \Sigma_{r_l \le j \le r_{l+1}-1} \; a_j \, [\, M_j(\hat{\xi}_j , \hat{\theta}_j) - \overline{M}(\hat{\theta}_j)\,] \,.$$

Let $\overline{\theta}$ be the linear interpolation of $\{ \hat{\theta}_j : r_l \le j \le r_{l+1} - 1 \}$, and $\overline{\xi}_j(\overline{\theta})$ the corresponding piecewise right continuous constant interpolation of $\{ \hat{\xi}_j : r_l \le j \le r_{l+1} - 1 \}$. For $\tau \in [\, t_{r_l} - t_n \,,\, t_{r_{l+1}} - t_n \,)$, define the following piecewise constant interpolations:

$$I_{1,n}(\tau) \equiv \Sigma_{r_l \le j \le r_{l+1}-1} \, (a_j / \delta_n) \, [\, M_j(\hat{\xi}_j , \hat{\theta}_j) - M_j(\overline{\xi}_j(\overline{\theta}) , \overline{\theta})\,] \,,$$
$$I_{2,n}(\tau) \equiv \Sigma_{r_l \le j \le r_{l+1}-1} \, (a_j / \delta_n) \, [\, \overline{M}(\overline{\theta}) - \overline{M}(\hat{\theta}_j)\,] \,,$$
$$I_{3,n}(\tau) \equiv \delta_n^{-1} \Sigma_{r_l \le j \le r_{l+1}-1} \, a_j \, [\, M_j(\overline{\xi}_j(\overline{\theta}) , \overline{\theta}) - \overline{M}(\overline{\theta})\,] \,.$$

Then $A_n(\tau) = I_{1,n}(\tau) + I_{2,n}(\tau) + I_{3,n}(\tau)$. Given A.6(b), we have $\sup_{0 \le t, t+s \le T} r_K (\int_{t \le \tau < t+s} I_{3,n}(\tau)\, d\tau \,, 0) \to 0$ as $n \to \infty$. By the continuity of M_j (A.2(b)) and $\overline{M}$ (A.6(a), and the compactness of Ξ, K, M_j and $\overline{M}$ are uniformly continuous ($-\, r_K$). Hence for any $\varepsilon > 0$, there exists $\eta_j > 0$, independent of $(\overline{\xi}_j(\overline{\theta}) , \overline{\theta})$, such that for all $(\xi , \theta) \in \Xi \times K$ with $r_\Xi(\xi , \overline{\xi}_j(\overline{\theta})) + r_K(\theta , \overline{\theta}) \le \eta_j$, we have $r_K(M_j(\xi , \theta) , M_j(\overline{\xi}_j(\overline{\theta}) , \overline{\theta})) \le \varepsilon / N_1$ for some large integer $N_1 > 0$. Also, there exists $\eta > 0$ independent of $\overline{\theta}$ such that for all $\theta \in K$ with $r_K(\theta , \overline{\theta}) \le \eta$, we have $r_K(\overline{M}(\theta) , \overline{M}(\overline{\theta})) \le \varepsilon / N_2$ for some large integer $N_2 > 0$. Let $\eta^n = \min \{ \eta \,, \eta_j \,;\, r_l \le j \le r_{l+1} - 1 \}$. Now the equicontinuity under the r_K–metric of $\{ \theta^n(\cdot) \}$ implies Assumption A.7(1), and the convergence of $\{ \theta^n(\cdot) \}$ under the r_K–metric implies A.7(2). Thus Assumption A.7 applies, and there exists a $\delta > 0$ with $\delta < \eta^n$ such that $r_K(\hat{\theta}_j , \overline{\theta}) \le \delta$ and $r_\Xi(\hat{\xi}_j , \overline{\xi}_j(\overline{\theta})) \le \eta^n - \delta$. Now since $\delta_n = \Sigma_{r_l \le j \le r_{l+1}-1} a_j$, and the convex combination operation is continuous under the r_K–metric, we obtain $r_K(I_{1,n}(\tau) , 0) \to 0$ and $r_K(I_{2,n}(\tau) , 0) \to 0$ as $n \to \infty$ $a.s. - I\!P$ for all $\tau \in [\, t \,,\, t + s \,)$. By Assumption A.2(a), we get

$$\sup_{0 \le t, t+s \le T} r_K (\int_{t \le \tau < t+s} I_{1,n}(\tau)\, d\tau \,, 0) \to 0 \,, \text{ and } \sup_{0 \le t, t+s \le T} r_K (\int_{t \le \tau < t+s} I_{2,n}(\tau)\, d\tau \,, 0) \to 0 \,.$$

Hence $\sup_{0 \le t, t+s \le T} r_K (\int_{t \le \tau < t+s} A_n(\tau)\, d\tau \,, 0) \to 0$ as $n \to \infty$ $a.s. - I\!P$. Substituting into (a.1), we get

$$(a.2) \quad \theta^n(t+s) - \theta^n(t) = \int_{t \le \tau < t+s} f_n(\tau)\, d\tau + o(1) \,, \quad \text{where } \sup_{0 \le t, t+s \le T} r_K(o(1) , 0) \to 0 \text{ as } n \to \infty \,.$$

Due to the uniform convergence of $\{ \theta^n(\cdot) \}$ in $[\, 0 \,, T \,]$ under the r_K–metric, the left hand side of (a.2) goes to $\theta(t+s) - \theta(t)$ in the r_K –metric uniformly in $[\,0\,,T\,]$ as $n \to \infty$. To complete the proof, we need only show that the right hand side of (a.2) goes to $\int_{t \le \tau < t+s} \overline{M}(\theta(\tau))\, d\tau$ in the r_K –metric uniformly in $[\,0\,,T\,]$ as $n \to \infty$. By Assumption A.2(a), it suffices to show $r_K\,(\,f_n(\tau)\,,\overline{M}(\theta(\tau))\,) \to 0$ as $n \to \infty$ for all $\tau \in [\,t\,,t+s\,)$, which in turn is implied by Assumption A.6(a), and the convergence of $r_K\,(\,\theta^n(\tau)\,,\theta(\tau)\,) \to 0$ as $n \to \infty$ for all $\tau \in [\,t\,,t+s\,)$. This completes (ii).

(iii) The proof is similar to BS's (1987) theorem 3.4, and KC's (1978) theorem 2.5.2.

PROOF OF THEOREM 4.1: (i) Let Ω' denote the union of the set where $\{\hat\theta_n\}$ is not in $I\!H$ with the union of the exceptional sets in A.3, B.6(c) and A.7. Fix $\omega \in \Omega - \Omega'$. Given $\sup_n \|\hat\theta_n\| \le \overline{B} < \infty$, and the definition of $\{\theta^n(\cdot)\}$, we get that $\{\theta^n(\cdot)\}$ is uniformly bounded in norm. Let $i_0 \equiv m(t_n+t)$ and $i_s \equiv m(t_n+t+s) - 1$. By the definition of θ^n, we have $\|\theta^n(t+s) - \theta^n(t)\| \le J_1 + J_2 + J_3$, where

$$J_1 \equiv \Sigma_{i_0+1 \le j \le i_s-1}\, a_j \,\|P_{k(j)}\, M_j(\hat\xi_j\,,\hat\theta_j)\|,$$
$$J_2 \equiv \alpha\, a_{i_0} \,\|P_{k(i_0)}\, M_{i_0}(\hat\xi_{i_0}\,,\hat\theta_{i_0})\|,$$
$$J_3 \equiv \beta\, a_{i_s} \,\|P_{k(i_s)}\, M_{i_s}(\hat\xi_{i_s}\,,\hat\theta_{i_s})\|,$$

for some $\alpha\,,\beta \in [0,1]$. Again for any finite $s\,,t > 0$, A.3 implies $i_s - i_0 < \infty$ uniformly in n. Since uniformly continuous functions map bounded sets into bounded sets, and the finite union of bounded sets is still bounded, B.2(a) and $\sup_n \|\hat\theta_n\| \le \overline{B} < \infty$ and A.3 imply $\|\theta^n(t+s) - \theta^n(t)\| \le D\,(\,s + \alpha\, a_{i_0} + \beta\, a_{i_s}\,) \to 0$ as $s \to 0$ for all n large enough, where

$$D = \sup_{i_0 \le j \le i_s} \sup_{\xi \in \Xi\,,\,\|\theta\| \le \overline{B}} \|M_j(\xi\,,\theta)\|.$$

Hence $\{\theta^n(\cdot)\}$ is equicontinuous under the norm topology.

(ii) Since for each $h \in I\!H$ and any $0 < t < \infty$,

$$|\,<\theta^n(t)\,,h>\,| \le \|\theta^n(t)\|\|h\|,$$

and

$$|\,<\theta^n(t+s) - \theta^n(t)\,,h>\,| \le \|\theta^n(t+s) - \theta^n(t)\|\|h\| \to 0 \quad \text{uniformly in } n \quad \text{as } s \to 0\,,$$

we have that $\{\theta^n(\cdot)\}$ is also uniformly bounded and equicontinuous under the weak topology. Since every set in $I\!H$ of the form $\{\theta : \|\theta\| \le B < \infty\}$ is compact under the weak topology, we can apply

Ascoli - Arzela's lemma to conclude that $\{ \theta^n(\cdot) \}$ is sequentially compact. Thus, every sequence of $\{ \theta^n(\cdot) \}$ has a convergent subsequence in the weak topology. For notational simplicity, we denote the convergent subsequence again by $\{ \theta^n(\cdot) \}$ and the weak limit by $\theta(\cdot)$, i.e., $< \theta^n(t) , h > \to < \theta(t) , h >$ as $n \to \infty$, for each $h \in \mathbb{H}$. Moreover, the convergence is uniform on each compact $t -$ set. Fix any $0 < T < \infty$ and let $t , t+s \in [0 , T]$. For each $h \in \mathbb{H}$ and for this convergent subsequence $\{ n \}$, we have

$$(a.3) \;\; < \theta^n(t+s) - \theta^n(t) , h > \; = \; < \Sigma_{m(t_n+t) \le j \le m(t_n+t+s)-1} \, a_j \, P_{k(j)} \, M_j(\hat{\xi}_j , \hat{\theta}_j) , h > - \, o(1) ,$$

where $\sup_{0 \le t, t+s \le T} | < o(1) , h > | \to 0$ as $n \to \infty$. We chose a sequence $\{ \delta_n > 0 \}$ such that

$$\delta_n \downarrow 0 \quad \text{and} \quad \delta_n^{-1} \sup_{j \ge n} a_j \to 0 \quad \text{as} \quad n \to \infty .$$

we select an increasing integer - valued sequence $\{ r_l \}$ with $r_1 = n$ such that $\Sigma_{r_l \le j \le r_{l+1}-1} \, a_j = \delta_n$, modulo an end value. Hence $\delta_n^{-1} (t_{r_{l+1}} - t_{r_l}) \to 1$ as $n \to \infty$. Let $0 < \delta_n < 2 \, \delta_n < \cdots \le t + s$ be a partition of $[0 , t + s]$ with equal length of subinterval δ_n.

$$\Sigma_{m(t_n+t) \le j \le m(t_n+t+s)-1} \, a_j \, P_{k(j)} \, M_j(\hat{\xi}_j , \hat{\theta}_j)$$

$$= \Sigma_l \, \delta_n \times [\, \delta_n^{-1} \, \Sigma_{r_l \le j \le r_{l+1}-1} \;\; a_j \, P_{k(j)} \, M_j(\hat{\xi}_j , \hat{\theta}_j) \,]$$

$$= \Sigma_l \, \delta_n \, f_n(\tau) + \Sigma_l \, \delta_n \, A_n(\tau) = \int_{t \le \tau < t+s} f_n(\tau) \, d\tau + \int_{t \le \tau < t+s} A_n(\tau) \, d\tau ,$$

where $f_n(\cdot)$ and $A_n(\cdot)$ are the piecewise right continuous constant interpolations for $\tau \in [\, t_{r_l} - t_n , \, t_{r_{l+1}} - t_n \,)$,

$$f_n(\tau) \equiv \delta_n^{-1} \, \Sigma_{r_l \le j \le r_{l+1}-1} \;\; a_j \, P_{k(j)} \, \overline{M}(\hat{\theta}_j)$$

and

$$A_n(\tau) \equiv \delta_n^{-1} \, \Sigma_{r_l \le j \le r_{l+1}-1} \;\; a_j \, P_{k(j)} \, [\, M_j(\hat{\xi}_j , \hat{\theta}_j) - \overline{M}(\hat{\theta}_j) \,] .$$

The equicontinuity (under the weak topology) of $\{ \theta^n(\cdot) \}$ implies Assumption A.7(1); and the convergence of $\{ \theta^n(\cdot) \}$ (under the weak topology) implies A.7(2). Thus Assumptions B.2(b), B.6(b)&(c), and A.7 imply that for each $h \in \mathbb{H}$,

$$\sup_{0 \le t, t+s \le T} | < \int_{t \le \tau < t+s} A_n(\tau) \, d\tau , h > | \to 0 \quad \text{as} \quad n \to \infty \qquad a.s. - \mathbb{P} .$$

Substituting into (a.3), we get that for each $h \in \mathbb{H}$,

$$(a.4) \quad < \theta^n(t+s) - \theta^n(t) \, , h > \; = \; < \int_{t \le \tau < t+s} f_n(\tau) \, d\tau \, , h > + \, o(1) \, ,$$

where $\sup_{0 \le t, t+s \le T} | o(1) | \to 0$ as $n \to \infty$. Due to the uniform convergence of $\{ \theta^n(\cdot) \}$ in $[0 , T]$ under the weak topology, the left hand side of (a.4) goes to $< \theta(t+s) - \theta(t) \, , h >$ as $n \to \infty$ for each $h \in \mathbb{H}$. To complete the proof, we need only show that the right hand side of (a.4) goes to $< \int_{t \le \tau < t+s} \overline{M}(\theta(\tau)) \, d\tau \, , h >$ as $n \to \infty$ for each $h \in \mathbb{H}$, and it suffices to show $\| f_n(\tau) - \overline{M}(\theta(\tau)) \| \to 0$ as $n \to \infty$ for each $\tau \in [t , t+s)$. Fix $\tau \in [t_{r_l} - t_n , t_{r_{l+1}} - t_n)$, and write

$$\delta_n^{-1} \Sigma_{r_l \le j \le r_{l+1}-1} \, a_j \, P_{k(j)} \, \overline{M}(\hat{\theta}_j) = b_{1n} + b_{2n} + b_{3n} + b_{4n} + b_{5n} \, ,$$

where

$$\begin{aligned} b_{1n} &\equiv \delta_n^{-1} \Sigma_{r_l \le j \le r_{l+1}-1} \, a_j \, \overline{M}(\theta(\tau)) \, , \\ b_{2n} &\equiv \delta_n^{-1} \Sigma_{r_l \le j \le r_{l+1}-1} \, a_j \, [\, P_{k(r_l)} \, \overline{M}(\theta(\tau)) - \overline{M}(\theta(\tau)) \,] \, , \\ b_{3n} &\equiv \delta_n^{-1} \Sigma_{r_l \le j \le r_{l+1}-1} \, a_j \, [\, (P_{k(j)} - P_{k(r_l)}) \, \overline{M}(\theta(\tau)) \,] \, , \\ b_{4n} &\equiv \delta_n^{-1} \Sigma_{r_l \le j \le r_{l+1}-1} \, a_j \, P_{k(j)} \, [\, \overline{M}(\hat{\theta}_{r_l}) - \overline{M}(\theta(\tau)) \,] \, , \\ b_{5n} &\equiv \delta_n^{-1} \Sigma_{r_l \le j \le r_{l+1}-1} \, a_j \, P_{k(j)} \, [\, \overline{M}(\hat{\theta}_j) - \overline{M}(\hat{\theta}_{r_l}) \,] \, . \end{aligned}$$

Due to the uniform boundedness of $\{ \hat{\theta}_n \}$ and Assumption B.2(a), as $n \to \infty$,

$$\begin{aligned} \max_{r_l \le j \le r_{l+1}} \| \hat{\theta}_j - \hat{\theta}_{r_l} \| &\le \max_{r_l \le j \le r_{l+1}} \| \Sigma_{r_l \le i \le j} \, a_i \, P_{k(i)} \, M_i(\hat{\xi}_i \, , \hat{\theta}_i) \| \\ &\le \Sigma_{r_l \le i \le r_{l+1}} \, a_i \, \sup_{r_l \le i \le r_{l+1}} \| P_{k(i)} \, M_i(\hat{\xi}_i \, , \hat{\theta}_i) \| \le D \times \delta_n \to 0 \, . \end{aligned}$$

By the triangle inequality, Assumption B.6(a), and the fact that $\| P_k \, h - h \| \to 0$ as $k \to \infty$ for any fixed $h \in \mathbb{H}$, we get $\| f_n(\tau) - \overline{M}(\theta(\tau)) \| \to 0$ as $n \to \infty$ for each $\tau \in [t , t+s)$. (The proof is the same as Yin's (1992, pp. 371-372).) Hence the right hand side of (A.4) goes to $< \int_{t \le \tau < t+s} \overline{M}(\theta(\tau)) \, d\tau \, , h >$ as $n \to \infty$ for each $h \in \mathbb{H}$. This completes (ii).

(iii) The proof is the same as that for Corollary 3.2, and is also similar to Schwartz & Berman's (1987) theorem 3.4, and KC's (1978) theorem 2.5.2.

PROOF OF THEOREMS 4.3 & 4.4: Since C.2(b) implies B.2(b), C.6(c) implies B.6(c), and C.7 implies A.7, it suffices to show that $\sup_n \| \hat{\theta}_n \| < \infty$, which in turn is equivalent to verifying the following Lemma A.1, an extension of Yin's lemma 1 (1992, p368) to the feedback case.

LEMMA A.1: Given Assumptions A.1, A.3, A.4, B.5, C.6, C.8 and C.9 for the Hilbert-valued TRMFP, if either (i) C.2 & A.7 or (ii) B.2(a) & C.7 hold, then $\lim_n T(n) = T < \infty \quad a.s.-I\!P$.

Proof: Suppose the conclusion does not hold. Since $\| P_{k(j)} \overline{\theta} \| \le \| \overline{\theta} \| < B$ for all j , $\{ \hat{\theta}_n \}$ will cross the sphere $\{ \theta : \| \theta \| = B \}$ infinitely often. Given C.8 and C.9, $V(\Theta^*)$ is a compact set. C.9(d) ensures that $[V(\overline{\theta}) , b] - [V(\overline{\theta}) , b] \cap V(\Theta^*)$ is a compact set with nonempty interior. Hence there exist positive reals δ_1 and δ_2 with $[\delta_1 , \delta_2] \subset (V(\overline{\theta}) , b]$ and $\delta_1 \overline{\in} V(\Theta^*)$. Since $\| P_{k(n)} - I \| \to 0$ as $n \to \infty$, for any $\eta_1 > 0$, there exists $K_{\eta_1} > 0$ such that for all $j \ge K_{\eta_1}$, we have $\| P_{k(j-1)} \theta - \theta \| \le \eta_1$ for all $\theta \in I\!H$. Define

$$\mu \equiv \min [j \ge K_{\eta_1} : V(\hat{\theta}_j) \ge \delta_1] ; \quad \nu \equiv \min [j \ge K_{\eta_1} : V(\hat{\theta}_j) \ge \delta_2] .$$

By definition, $\mu < \nu < \infty$ almost surely. Then $V(\hat{\theta}_j) \in [\delta_1 , \delta_2]$ for $j \in [\mu , \nu]$. Hence the set $\{ \hat{\theta}_j : \mu-1 \le j \le \nu \}$ is bounded, in particular, for all $j \in [\mu-1 , \nu]$, $\| \hat{\theta}_j \| \le B < B_1$ (Suppose there exists an $j \in [\mu-1 , \nu]$ with $\| \hat{\theta}_j \| > B$, then $V(\hat{\theta}_j) > b \ge \delta_2 > \delta_1$, a contradiction. Choose a small $\eta > 0$ and a positive function ε such that $\varepsilon(\eta) \to 0$ as $\eta \to 0$. Fix $\eta_1 = \varepsilon(\eta)$. Define $t_0 = 0$, $t_n = \Sigma_{0 \le i \le n-1} a_i$ for $n > 0$, and $m(i,t) \equiv \max [n > i : t_n - t_i \le t]$ if $t \ge 0$ and $m(i,t) \equiv 0$ if $t < 0$. By definition of μ and the choice of η ,

$$\mu - 1 < m(\mu-1 , \eta) \le \nu , \quad \delta_1 \le V(\hat{\theta}_{m(\mu-1,\eta)}) \le \delta_2 .$$

Now we show that $V(\hat{\theta}_{m(\mu-1,\eta)}) < \delta_1$ to conclude the proof. By Taylor expansion, we have

$$\begin{aligned} V(\hat{\theta}_{m(\mu-1,\eta)}) - V(\hat{\theta}_{\mu-1}) = &< V'(\hat{\theta}_{\mu-1}) , [\hat{\theta}_{m(\mu-1,\eta)} - \hat{\theta}_{\mu-1}] > \\ &+ R_2(\hat{\theta}_{\mu-1} , \hat{\theta}_{m(\mu-1,\eta)} - \hat{\theta}_{\mu-1}) , \end{aligned}$$

where

$$\| R_2(\hat{\theta}_{\mu-1} , \hat{\theta}_{m(\mu-1,\eta)} - \hat{\theta}_{\mu-1}) \| \le \text{constant} \times \| \hat{\theta}_{m(\mu-1,\eta)} - \hat{\theta}_{\mu-1} \|^2 .$$

Since $\{ \hat{\theta}_j : \mu-1 \le j \le \nu \}$ is bounded by B , there is no truncation from $\hat{\theta}_{\mu-1}$ to $\hat{\theta}_{m(\mu-1,\eta)}$, and we have

$$\| \hat{\theta}_{m(\mu-1,\eta)} - \hat{\theta}_{\mu-1} \| = \| \Sigma_{\mu-1 \le j \le m(\mu-1,\eta)-1} \, a_j \, P_{k(j)} \, M_j(\hat{\xi}_j , \hat{\theta}_j) \| ,$$

and

$$< V'(\hat{\theta}_{\mu-1}) \, , \hat{\theta}_{m(\mu-1,\eta)} - \hat{\theta}_{\mu-1} > \; = \Sigma_{\mu-1 \le j \le m(\mu-1,\eta)-1} \, a_j < V'(\hat{\theta}_{\mu-1}) \, , P_{k(j)} \, M_j(\hat{\xi}_j \, , \hat{\theta}_j) > \; = I_1 + I_2 + I_3 + I_4 \, ,$$

where

$$I_1 \equiv \Sigma_{\mu-1 \le j \le m(\mu-1,\eta)-1} \, a_j < V'(\hat{\theta}_{\mu-1}) \, , \overline{M}(\hat{\theta}_{\mu-1}) > \, ,$$
$$I_2 \equiv \Sigma_{\mu-1 \le j \le m(\mu-1,\eta)-1} \, a_j < V'(\hat{\theta}_{\mu-1}) \, , [\, P_{k(j)} \, \overline{M}(\hat{\theta}_{\mu-1}) - \overline{M}(\hat{\theta}_{\mu-1}) \,] > \, ,$$
$$I_3 \equiv < V'(\hat{\theta}_{\mu-1}) \, , \Sigma_{\mu-1 \le j \le m(\mu-1,\eta)-1} \, a_j \, P_{k(j)} \, [\, M_j(\xi_j(\hat{\theta}_{\mu-1}) \, , \hat{\theta}_{\mu-1}) - \overline{M}(\hat{\theta}_{\mu-1}) \,] > \, ,$$
$$I_4 \equiv < V'(\hat{\theta}_{\mu-1}) \, , \Sigma_{\mu-1 \le j \le m(\mu-1,\eta)-1} \, a_j \, P_{k(j)} \, [\, M_j(\hat{\xi}_j \, , \hat{\theta}_j) - M_j(\xi_j(\hat{\theta}_{\mu-1}) \, , \hat{\theta}_{\mu-1}) \,] > .$$

Since $\{ \hat{\theta}_j : \mu-1 \le j \le \nu \}$ is bounded by B, $\hat{\xi}_j \in \Xi$, a bounded set, B.2(a) (i.e., C.2(a)) and because a finite union of bounded sets is a bounded set we have that $\sup_{\mu-1 \le j \le m(\mu-1,\eta)-1} \| M_j(\hat{\xi}_j \, , \hat{\theta}_j) \| < \infty$, given the definition of $m(\mu-1,\eta)$, so that $\| \hat{\theta}_{m(\mu-1,\eta)} - \hat{\theta}_{\mu-1} \| = O(\eta)$. Now $V'(\hat{\theta}_{\mu-1})$ is compact (hence bounded) (see, e.g., Berger, p. 91, (2.4.6)); we have $\| I_2 \| \le \text{constant} \times \eta \times \eta_1 = o(\eta)$ by C.6(a) and a property of $P_{k(j)}$; and $\| I_3 \| = o(\eta)$ by C.6(c). Similiar to the proof of $\| \hat{\theta}_{m(\mu-1,\eta)} - \hat{\theta}_{\mu-1} \| = O(\eta)$, we have $\| \hat{\theta}_j - \hat{\theta}_{\mu-1} \| = O(\eta)$, (hence $d(\hat{\theta}_j \, , \hat{\theta}_{\mu-1}) = O(\eta)$), for any $j \in [\, \mu-1 \, , m(\mu-1,\eta)-1 \,]$. This together with either C.2(b) & A.7 or B.2(a) & C.7 implies that $\| I_4 \| = o(\eta)$. Now C.9(c) implies $V(\hat{\theta}_{m(\mu-1,\eta)}) - V(\hat{\theta}_{\mu-1}) = - \mid O(\eta) \mid + o(\eta) + O(\eta^2)$. Since $V(\hat{\theta}_{\mu-1}) < \delta_1$ by definition of μ, we get $V(\hat{\theta}_{m(\mu-1,\eta)}) < \delta_1$, thus completing the proof.

PROOF OF LEMMA 5.2: Similiar to the proof of theorem A.1 in KW (1994), but apply the strong law of large numbers of corollary 3.8 in CW (1994).

PROOF OF LEMMA 5.3: Similiar to the proof of lemma A.2 in KW (1994), but using different metrics.

LEMMA A.2: Given Assumptions D.1, D.4(a)&(c), recursively define $\xi_{n+1}(\theta) = \rho_n(\xi_n(\theta) \, , \theta \, , Z_{n+1})$ for any constant sequence $\{\theta\}$ with $\theta \in I\!H$ and all $n = 0,1,2,...$. Then for any $\| \theta \| \le \overline{B} < \infty$, $\{ \xi_n(\theta) \, ; n = 0,1,2,... \}$ is a sequence of bounded $L_2(\Xi)$ –NED functions on $\{ D_n \}$ of size $-1/2$.

Proof: This is a Hilbert space version of KW's (1994) proposition 4.4. Their proof goes through here after proper translation.

PROOF OF LEMMA 5.5: For any fixed $\theta \in I\!H$ with $\| \theta \| \le \overline{B}$, Given Assumptions D.1, D.4(a)&(c), $\{ \xi_n(\theta) \, ; n = 0,1,2,... \}$ is a sequence of bounded $L_2(\Xi)$ –NED functions on $\{ D_n \}$ of size $-1/2$. By the minimun mean squared error property and Assumption D.2(b), we have

$$\| P_{k(n)} M_n(\xi_n(\theta), \theta) - E[P_{k(n)} M_n(\xi_n(\theta), \theta) \mid F_{n-m}^{n+m}] \|_2$$
$$\le \| P_{k(n)} M_n(\xi_n(\theta), \theta) - P_{k(n)} M_n(E[\xi_n(\theta) \mid F_{n-m}^{n+m}], \theta) \|_2$$
$$\le c_{3,n} \| \xi_n(\theta) - E[\xi_n(\theta) \mid F_{n-m}^{n+m}] \|_2 .$$

Hence for each fixed θ with $\|\theta\| \le \overline{B}$, $\{ P_{k(n)} M_n(\xi_n(\theta), \theta) ; n = 0,1,2,... \}$ is a sequence of $L_2(I\!H)$ - NED functions on $\{D_n\}$ with $c_n = O(c_{3,n})$ and size $-1/2$. Now CW's (1994) lemma 4.3 implies that, for each $\theta \in I\!H$, $\{ P_{k(n)} M_n(\xi_n(\theta), \theta) - E[P_{k(n)} M_n(\xi_n(\theta), \theta)] ; n = 0,1,... \}$ is a $L_2(I\!H) -$ mixingale with $c_n = O(c_{3,n})$ and size $-1/2$. Hence Assumption D.6(c) is satisfied given D.3.

PROOF OF COROLLARY 5.6: This directly follows from Theorem 4.4, Lemmas 5.2, 5.3 and 5.5.

PROOF OF PROPOSITION 6.1: It is obvious that Assumption D.1(a) is satisfied by $G = [-1,1] \times [l_1, l_2]$, and D.1(b) is satisfied given (6.1.2) and the conditions on $\{ u_t \}$. Given the conditions, we have that Ξ is a compact subset of $I\!R^2$. For each t, any $(\xi, \theta), (\overline{\xi}, \overline{\theta}) \in \Xi \times I\!H$ with $\|\theta\| \le B$, $\|\overline{\theta}\| \le B$, the triangle inequality gives $\| M_t(\xi, \theta) - M_t(\overline{\xi}, \overline{\theta}) \| \le I_1 + I_2$, where

$$I_1 \equiv \| [h_t]^{-1} K((\cdot - \xi_1)/h_t) [(\xi_2 - \theta) - (\overline{\xi}_2 - \overline{\theta})] \| ;$$
$$I_2 \equiv \| [h_t]^{-1} [K((\cdot - \xi_1)/h_t) - K((\cdot - \overline{\xi}_1)/h_t)] \ (\overline{\xi}_2 - \overline{\theta}) \| .$$

Now

$$I_1{}^2 \le \| [h_t]^{-1} K((\cdot - \xi_1)/h_t) \|^2 \| (\xi_2 - \overline{\xi}_2) - (\theta - \overline{\theta}) \|^2$$
$$= [h_t]^{-1} \int_{[-h_t^{-1}, h_t^{-1}]} [K(y - \xi_1 [h_t]^{-1})]^2 dy \times \| (\xi_2 - \overline{\xi}_2) - (\theta - \overline{\theta}) \|^2$$
$$\le [h_t]^{-1} C \| (\xi_2 - \overline{\xi}_2) - (\theta - \overline{\theta}) \|^2 ,$$

where the last inequality is due to (6.1.3). By the triangle inequality, we get $I_1 \le [h_t]^{-1/2} C_1 [\, | \xi_2 - \overline{\xi}_2 | + \| \theta - \overline{\theta} \|]$.

$$I_2{}^2 \le \| [h_t]^{-1} [K((\cdot - \xi_1)/h_t) - K((\cdot - \overline{\xi}_1)/h_t)] \|^2 \times \| \overline{\xi}_2 - \overline{\theta} \|^2$$
$$\le [h_t]^{-4} [C_2]^2 \, | \xi_1 - \overline{\xi}_1 |^2 \times [C_3 + B]^2 .$$

Hence $I_2 \le [h_t]^{-2} C_2 \, | \xi_1 - \overline{\xi}_1 | \, [C_3 + B]$. We get

$$\| M_t(\xi, \theta) - M_t(\overline{\xi}, \overline{\theta}) \| \le \max([h_t]^{-1/2} C_1 , [h_t]^{-2} C_2 [C_3 + B] , 1) \times [\| \xi - \overline{\xi} \|_{\Xi} + \| \theta - \overline{\theta} \|] .$$

So for each t, M_t is Lipschitz continuous on $\Xi \times \{ \theta : \|\theta\| \le B \}$, thus it is uniformly continuous on

$\Xi \times \{ \theta : \|\theta\| \le B \}$. From the form of M_t, it is obvious that it is weakly sequentially continous, hence Assumption D.2(a) (i.e., B.2) is satisfied. From the above proof, we also have: for any $\xi , \bar{\xi} \in \Xi$, $\|\theta\| \le B$,

$$\| M_t(\xi , \theta) - M_t(\bar{\xi} , \theta)\| \le \max ([h_t]^{-1/2} C_1 , [h_t]^{-2} C_2 [C_3 + B]) \times \|\xi - \bar{\xi} \|_{\Xi} .$$

Given (6.1.4), we can set $c_{3,t} = O([h_t]^{-2})$, and Assumption D.2(b) is satisfied. Assumption D.3 is satisfied given (6.1.4) and $a_t = (t+1)^{-1}$. Since $\rho_t'(\xi,\theta,z) = (\rho_{1,t} , \rho_{2,t})$, $\rho_{1,t}(\xi,\theta,z) = z_1$, and $\rho_{2,t}(\xi,\theta,z) = a(z_1) + b\,\theta(z_1) + z_2$, Assumptions D.4(a)&(b)&(c) are satisfied with $c_0 = 0$, $c_1 = b$, and $c_3 = \max(c_a , c_\theta , 1)$ respectively, where c_a is the Lipschitz constant for function a and c_θ the Lipschitz constant for Lipschitz continuous function θ. Note that given conditions (6.1.3), (6.1.6), and the definition of $\{ \hat{\theta}_t \}$, we have the Lipschitz continuity of θ in $\rho(\xi,\theta,z)$. Assumption B.5 is directly assumed by (6.1.6). Since $\{ X_t \}$ and $\{ u_t \}$ are independent and $E[u_t] = 0$, we have

$$\begin{aligned} E[M_t(\xi_t(\theta) , \theta(\cdot))] &= E[(h_t)^{-1} K((\cdot - \xi_{1,t}) / h_t) (\xi_{2,t} - \theta(\cdot))] \\ &= E[(h_t)^{-1} K((\cdot - X_t) / h_t) [a(X_t) + b\,\theta(X_t) - \theta(\cdot)]] \\ &= \int_{[-1,1]} (h_t)^{-1} K((\cdot - y) / h_t) [a(y) + b\,\theta(y) - \theta(\cdot)] f(y)\, dy \\ &= \int K(\eta) [a(\cdot + \eta\, h_t) + b\,\theta(\cdot + \eta\, h_t) - \theta(\cdot)] f(\cdot + \eta\, h_t)\, d\eta . \end{aligned}$$

Under Conditions (6.1.1), (6.1.3) - (6.1.6), and the construction of $\{ \hat{\theta}_t \}$, theorem 2.1.1 in Rao (1983) implies

$$\lim_{t \to \infty} \sup_{x \in [-1,1]} | E [M_t(\xi_t(\theta) , \theta(x))] - [a(x) + b\,\theta(x) - \theta(x)] f(x) | = 0 .$$

Hence

$$\lim_{t \to \infty} \| E[M_t(\xi_t(\theta) , \theta(\cdot))] - [a(\cdot) - (1 - b)\, \theta(\cdot)] f(\cdot) \|^2 = 0 .$$

Set $\overline{M}(\theta(\cdot)) \equiv [a(\cdot) - (1 - b)\, \theta(\cdot)] f(\cdot)$; then Assumption D.6(a) is satisfied. Moreover, C.8 is satisfied with $\Theta^* = \{ \theta \in L_2(I\!R) : \overline{M}(\theta) = 0 \} = \{ \theta_o = a(\cdot) / (1 - b) \}$. Define a functional $V : L_2(I\!R) \to I\!R$ as $V(\theta) = \|\theta - \theta_o\|^2$. It is easy to see that Assumptions C.9 (a),(b)&(d) are satisfied, since with $f > 0$ *a.e.*, $1 - b > 0$, we have

$$< V'(\theta) , \overline{M}(\theta) > = < 2 [\theta - \theta_o] , f [1-b] [\theta_o - \theta] > < 0 , \quad \text{for all } \theta \ne \theta_o ;$$

hence C.9(c) is satisfied. The result now follows from Corollary 5.6 and Remark 4.5.

PROOF OF PROPOSITION 6.2: Assumption D.1 is directly given by (6.2.1). By definition, Ξ is bounded and $\|\hat{\theta}_t\| \equiv 1$. For each (ξ, θ) and $(\bar{\xi}, \bar{\theta})$, we have

$$M_t(\xi, \theta(\cdot)) - M_t(\bar{\xi}, \bar{\theta}(\cdot)) = I_1 + I_2 + I_3 + I_4 + I_5 ,$$

where

$$I_1 \equiv Q(\xi)\,[h_t]^{-1}\,[\,K((\cdot - \xi_1)/h_t) - K((\cdot - \bar{\xi}_1)/h_t)\,;$$
$$I_2 \equiv [\,Q(\xi) - Q(\bar{\xi})\,]\,[h_t]^{-1}\,K((\cdot - \bar{\xi}_1)/h_t)\,;$$
$$I_3 \equiv -[\,\theta(\cdot) - \bar{\theta}(\cdot)\,]\,Q(\xi)\,[h_t]^{-1}\int_{[0,1]} K((y - \xi_1)/h_t)\,dy\,;$$
$$I_4 \equiv -\bar{\theta}(\cdot)\,Q(\xi)\,[h_t]^{-1}\int_{[0,1]}[\,K((y - \xi_1)/h_t) - K((y - \bar{\xi}_1)/h_t)\,]\,dy\,;$$
$$I_5 \equiv -\bar{\theta}(\cdot)\,[\,Q(\xi) - Q(\bar{\xi})\,]\,[h_t]^{-1}\int_{[0,1]} K((y - \bar{\xi}_1)/h_t)\,dy\,.$$

Given (6.2.2) - (6.2.4), and $\|\bar{\theta}\| = 1$ by definition, we have

$$\|I_1\| \le |\,\xi_1 - \bar{\xi}_1\,| \times O([h_t]^{-2})\,; \qquad \|I_2\| \le \|\xi - \bar{\xi}\|_{\Xi} \times O([h_t]^{-1/2})\,;$$

$$\|I_3\| \le \|\theta - \bar{\theta}\| \times O(1)\,; \quad \|I_4\| \le O([h_t]^{-2})\,; \quad \text{and } \|I_5\| \le \|\xi - \bar{\xi}\|_{\Xi} \times O(1)\,.$$

Hence

$$\|M_t(\xi, \theta(\cdot)) - M_t(\bar{\xi}, \bar{\theta}(\cdot))\| \le O([h_t]^{-2}) \times [\,\|\xi - \bar{\xi}\|_{\Xi} + \|\theta - \bar{\theta}\|\,]\,,$$

so M_t is Lipschitz continuous for each t, thus Assumption D.2(a) is satisfied. It is also clear that D.2(b) is satisfied with $c_{3,t} = O([h_t]^{-2})$. Since by (6.2.2), (6.2.3),

$$(a_t)^{-1} = \int_{[0,1]} S_0(y)\,dy + \sum_{j=0}^{t} Q(\xi_{1,j}, \xi_{2,j}) \int_{[0,1]} K((y - \xi_{1,j})/h_j)\,[h_j]^{-1}\,dy = O(t+1) \quad a.s. - I\!P\,,$$

Assumption D.3 is satisfied given (6.2.4). Since $\xi_{1,t} = x_t$ and $\xi_{2,t} = Z_t$, we have $I\!P\,[\,\xi_{1,t} \le x \mid \xi^{t-1}, \theta^t, Z_t\,] = \int_{[0,x]} \theta_t(y)\,dy$. We can define $\rho' = (\rho_1, \rho_2)$ with $\rho_2(\xi, \theta, z) = z$, and $I\!P\,[\rho_1(\xi, \theta, z) \le x\,] = \int_{[0,x]} \theta(y)\,dy$. Then Assumptions D.4(a)&(b)&(c) are satisfied with $c_0 = 0$, $c_1 = 1$, $c_2 = 1$ respectively. Assumption B.5 is directly assumed. For any $x \in [0,1]$ and for any fixed $\theta(\cdot)$, since x_t is independent of $\{Z_j\}$, we have $E[\,M_t(\xi_t, \theta(x))\,] =$

$$E[\,Q_2(Z_t)\,] \times (\,E[\,Q_1(x_t)\,K((x - x_t)/h_t)\,[h_t]^{-1}\,] - \theta(x) \int_{[0,1]} E[\,Q_1(x_t)\,K((y - x_t)/h_t)\,[h_t]^{-1}\,]\,dy\,)\,.$$

Since Q_1 and θ are continuous on $[0,1]$, given (6.2.3) and Rao's (1983) theorem 2.1.1, we have

$$\lim_{h_t \to 0} E[\,Q_1(x_t)\,K((x - x_t)/h_t)\,[h_t]^{-1}\,] =$$

$$\int_{[0,1]} Q_1(y)\, K(\,(\,x-y\,)/h_t\,)\,[h_t]^{-1}\,\theta(y)\,dy = Q_1(x)\,\theta(x) \quad \text{uniformly in } x \in [0,1]\,.$$

Since $E[\,Q_2(Z_t)\,]$ is the same for all t and $E[\,Q(\,\cdot\,,Z_t\,)\,] = Q_1(\cdot)\times E[\,Q_2(Z_t)\,]$, there exists a function $\overline{M} : L_2[0,1] \to L_2[0,1]$

$$\overline{M}(\theta(\cdot)) = \theta(\cdot)\,E[\,Q(\cdot,Z_t) - \int_{[0,1]} Q(s,Z_t)\,\theta(s)\,ds\,]\,,$$

such that

$$\lim_{t\to\infty} \|\,E[\,M_t(\,\xi_t(\,\theta\,)\,,\theta(\,\cdot\,)\,)\,] - \overline{M}(\,\theta(\,\cdot\,)\,)\| = 0\,,$$

hence Assumption D.6(a) is satisfied. D.6(b) is clearly satisfied. Now Corollary 5.6 and $\sup\|\hat{\theta}_t\| = 1$ implies that $\hat{\theta}_t \to \Theta^*$ almost surely in the weak topology as $t \to \infty$.

PROOF OF PROPOSITION 6.3: Assumption D.1 is directly assumed by (6.3.3). Given $M_t = \xi_t(\theta) - \theta$, D.2(a)&(b) are clearly satisfied with $c_{3,t} = 1$. Assumption D.3 is then satisfied with $a_t = (\,t+1\,)^{-1}$. Given (6.3.2) and $\hat{\theta}_{t+1} = \hat{\theta}_t + (t+1)^{-1}\,[\,\hat{\xi}_t - \hat{\theta}_t\,]$, to show $\{\hat{\theta}_t\}$ is an increasing concave sequence almost surely, it suffices to show that :

$\hat{\xi}_t(s) = \beta \max\,[\,U(c_t) + \hat{\theta}_t(\,Z_t\,s - c_t\,)\,]$ is increasing and concave almost surely, given that both $U(\cdot)$ and $\hat{\theta}_t(\cdot)$ are increasing and concave. Given (6.3.1) - (6.3.3), $\hat{\xi}_t(s) = \beta\,[\,U(\,c_t^*\,) + \hat{\theta}_t(\,Z_t\,s - c_t^*\,)\,]$, where c_t^* solves the equation $U'(\,c_t^*\,) = \hat{\theta}_t'\,(\,Z_t\,s - c_t^*\,)$. Assumption D.4(a)&(b) are satisfied with $c_0 = 0$, $c_1 = \beta$. Since $d\xi\,/\,dZ = \beta\,s\,\theta'(\,Zs - c^*\,)$, Assumption D.4(c) is satisfied too. B.5 is directly assumed. Now for each fixed θ, $M_t\,(\,\xi_t(\theta)\,,\theta\,) = \xi_t(\theta) - \theta$. Given (6.3.3), we have $\{\xi_t(\theta)\}$ is an i.i.d sequence, $E[\,M_t\,(\,\xi_t(\theta)\,,\theta\,)\,] = E[\,\xi_t(\theta)\,] - \theta \to W - \theta$, where $W = \lim_{t\to\infty} E_t\,[\,\xi_t(\theta)\,]$ is defined as $W(s) = \beta\,E\,[\,U(\bar{c}_t) + \theta(\,Z_t\,s - \bar{c}_t\,)\,]$, and $U'(\,\bar{c}_t\,) = \theta'\,(\,Z_t\,s - \bar{c}_t\,)$. Hence we take $\overline{M}(\theta) \equiv W - \theta$. Given (6.3.1) and (6.3.3) and that θ belongs to the set of twice continuous differentiable, increasing, and concave functions, it is obvious that $\overline{M}$ satisfies D.6(a)&(b). Assumption C.8 is satisfied with $\Theta^* = \{\theta : \overline{M}(\theta) = 0\} = \{\,\theta = W\,\}$. C.9 is satisfied with $V(\theta) \equiv \|\theta - W\|^2$. Hence the result follows from Corollary 5.6.

PROOF OF PROPOSITION 6.4: Since Z_t is uniformly bounded in t, there exits G, a norm-bounded convex subset of $I\!R^2$, containing the support of Z_t for all t, so that Assumption D.1(a) is satisfied. Assumption D.1(b) is given by (6.4.2). For any $(\,\xi\,,\theta\,)\,,(\,\bar{\xi}\,,\bar{\theta}\,) \in \Xi \times I\!H$, we have

$$\| M_t(\xi , \theta) - M_t(\bar\xi , \bar\theta)\|^2 = \| M_{1,t}(\xi , \theta) - M_{1,t}(\bar\xi , \bar\theta)\|^2 + \| M_{2,t}(\xi , \theta) - M_{2,t}(\bar\xi , \bar\theta)\|^2 \equiv I_1 + I_2 .$$

By the triangle inequality and then the Lipschitz continuity of K_1 assumed in (6.4.3), we get

$$\begin{aligned} I_1 &\le [h_{1,t}]^{-2} \| K_1((\cdot - \xi_2) / h_{1,t}) - K_1((\cdot - \bar\xi_2) / h_{1,t})\|^2 + \|\theta_1 - \bar\theta_1 \|^2 \\ &= [h_{1,t}]^{-2} \int_{[0,1]} [K_1((x - \xi_2)/h_{1,t}) - K_1((x - \bar\xi_2)/h_{1,t})]^2 dx + \|\theta_1 - \bar\theta_1 \|^2 \\ &\le [h_{1,t}]^{-4} C_1^2 \, | \xi_2 - \bar\xi_2 |^2 + \|\theta_1 - \bar\theta_1 \|^2 . \end{aligned}$$

Similiarly we get

$$I_2 \le [h_{2,t}]^{-4} C_2^2 \, | \xi_1 - \bar\xi_1 |^2 + \|\theta_2 - \bar\theta_2 \|^2 .$$

Denote $c_{3,t} = \max (C_1 [h_{1,t}]^{-2} , C_2 [h_{2,t}]^{-2})$; then

$$\| M_t(\xi , \theta) - M_t(\bar\xi , \bar\theta)\|^2 \le c_{3,t}^2 \|\xi - \bar\xi \|_\Xi^2 + \|\theta - \bar\theta \|^2 .$$

Hence

$$\| M_t(\xi , \theta) - M_t(\bar\xi , \bar\theta)\| \le \max (c_{3,t} , 1) [\|\xi - \bar\xi \|_\Xi^2 + \|\theta - \bar\theta \|^2]^{1/2} .$$

Hence for each t , the mapping $M_t : \Xi \times \mathbb{H} \to \mathbb{H}$ is Lipschitz continuous, thus is uniformly continous. Given the form of M_t , it is clearly weakly sequentially continuous, too. Thus, Assumption D.2(a) (i.e.(B.2)) is satisfied. Now for each fixed $\theta \in \mathbb{H}$, take any $\xi , \bar\xi \in \Xi$; we also get from the above that $\| M_t(\xi , \theta) - M_t(\bar\xi , \theta)\| \le c_{3,t} \|\xi - \bar\xi \|_\Xi$, and we have that Assumption D.2(b) is satisfied. Since $a_t = (t+1)^{-1}$, given (6.4.4), $c_{3,t}\, a_t = O((t+1)^{2\delta-1})$ for some $0 < \delta < 1/4$, thus Assumption D.3 is satisfied. Since $\rho_t(\xi,\theta,z) = \int_{[0,1]} x\, \theta(x)\, dx - z$, Assumptions D.4(a, b, c) are satisfied with $c_0 = 0$, $c_1 = 1$ and $c_2 = 1$ respectively. Assumption A.5 is directly assumed. For each fixed $\theta \in L_2([0,1]) \times L_2([0,1])$,

$$E [M_{1,t}(\xi_t(\theta) , \theta)] = E [(h_{1,t})^{-1} K_1((\cdot - \xi_{2,t}(\theta)) / h_{1,t})] - \theta_{12} ,$$

and

$$E [M_{2,t}(\xi_t(\theta) , \theta)] = E [(h_{2,t})^{-1} K_2((\cdot - \xi_{1,t}(\theta)) / h_{2,t})] - \theta_{21} .$$

Under Conditions (6.4.1), (6.4.3) and (6.4.4), theorem 2.1.1 in Rao (1983) implies

$$\lim_{t\to\infty} \sup_{x\in[0,1]} | E [(h_{1,t})^{-1} K_1((x - \xi_{2,t}(\theta)) / h_{1,t})] - f_2(x) | = 0 .$$

Hence

$$\lim_{t\to\infty} \| E [M_{1,t}(\xi_t(\theta) , \theta)] - (f_2 - \theta_{12}) \|$$

$$= \left(\int_{[0,1]} \left[\lim_{t\to\infty} E\left[(h_{1,t})^{-1} K_1((x - \xi_{2,t}(\theta)) / h_{1,t}) \right] - f_2(x) \right]^2 dx \right)^{1/2} = 0 .$$

Hence $\lim_{t\to\infty} E[M_{1,t}(\xi_t(\theta) , \theta)] = f_2 - \theta_{12}$ in the norm topology. Similarly we have $\lim_{t\to\infty} E[M_{2,t}(\xi_t(\theta) , \theta)] = f_1 - \theta_{21}$ in norm. Thus Assumption D.6(a) is satisfied with

$$\overline{M}'(\theta) = (\overline{M}_1(\theta) , \overline{M}_2(\theta)) = (f_2 - \theta_{12} , f_1 - \theta_{21}) = (f_2 , f_1) - \theta' .$$

It is obvious that $\overline{M}$ satisfies D.6(b). Since

$$\Theta^* = \{ \theta \in L_2([0,1]) \times L_2([0,1]) : \overline{M}(\theta) = 0 \} = \{ \theta_o = (f_2 , f_1) \} ,$$

C.8 is satisfied. If we define a functional $V : L_2([0,1]) \times L_2([0,1]) \to I\!R$ as $V(\theta) = \| \theta - \theta_o \|^2$, again we see that C.9 is satisfied. Hence the result follows from Corollary 5.6 and Remark 4.5.

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