Voting and Information Aggregation

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THE FRAMEWORK

1. Finite Set of Agents
2. Binary Decision
3. Limited Information not Commonly Available
4. (Possibly) Different Preferences

THE QUESTION

How to arrive at a good decision?
This week I focus on how well voting mechanisms aggregate information.
Keep in mind:

1. Mechanism Design Approach Possible Alternative
2. Welfare Objective Clear with Homogeneous Preferences

MODEL

1. N Players (usually assumed to be odd)
2. Two states: $X = 0, 1$.
3. Two actions a, $a = 0, 1$.
4. Preferences:
   Agent i has utility $u_i(a, X)$
   Typically assume: $u_i(j, j) = 0$, $u_i(0, 1) = -q_i$, $u_i(1, 0) = -(1 - q_i)$ for $q_i \in (0, 1)$.
5. Information: Agent i receives signal about X.
6. Strategy: Voting rule given information.

PREFERENCES

Voter i prefers outcome 0 if

$$-(1 - p_0)q_i \geq -p_0(1 - q_i)$$

or

$$p_0 \geq q_i.$$

$X = 0$ means guilt; $X = 1$ means innocent.
$a = 0$: convict; $a = 1$: free.
q_i is the standard of proof needed to convict:
Voter i prefers to convict only when probability of guilty, p_0 is at least q_i.
Note:

- Ex post heterogeneity ruled out.
- Ex ante heterogeneity permitted.

INFORMATION

Prior probability $\pi \in (0, 1)$ that state is 0.
Signal $P(1 \mid 1) = P(0 \mid 0) = p \in (\frac{1}{2}, 1)$
Individuals receive conditionally iid signals.
This means we are assuming:

1. Symmetry across states (not important)
2. Binary (possibly important)
3. Symmetry across individuals (sometimes important)
4. Conditionally iid signals (not explored, but potentially important)

Note, by law of large numbers, only strategic problems prevent large groups from learning almost everything by pooling their signals.

STRATEGIES

- Informative: $v_i(k) = k$.
  (Vote for Signal)
- Sincere: $v_i(k) = 0$ if and only if $Pr(X = 0 \mid k) > q_i$.
  (Vote for best option given signal.)
- Strategic: Nash Equilibrium (typically in undominated strategies)
  (Vote for best assuming pivotal.)

CONDORCET JURY THEOREM

Theorem

If all individuals vote informatively, then the probability that a majority votes for the better outcome is greater than p and converges to 1 as N goes to infinity.

Comments:

1. Informative voting means that everyone votes for better alternative with probability p.
2. Better outcome is well defined.
3. Independent Information.
4. No explicit motivation for voting behavior
5. Two Conclusions:
   1. The majority is better than an individual.
   2. Asymptotically the majority is right.

WHY IS JURY THEOREM TRUE?

The first part of the Jury Theorem follows from a routine argument.
Assume that the population is odd, say $N = 2n + 1$.
Let $P(n; N)$ be the probability that there are at least n votes out of N for the better outcome.
I will show that $P(n + 1; 2n + 1) > P(n; 2n - 1)$. Since $P(1; 1) = p$, the result will follow by induction. When you go from $2n - 1$ to $2n + 1$ you influence the outcome only when "the last two" votes are the same and the vote in the $2n - 1$ case is close. Further, it is more likely that the final two votes will reverse the outcome in favor of the better candidate than the worse one.

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Algebra

Note that P(n+1;2n+1) =

$$2p(1-p)P(n;2n-1)+p^2P(n-1;2n-1)+(1-p)^2P(n+1;2n-1).$$

This equation follows from dynamic programming logic. After $2n-1$ votes, the probability that the final two votes split is $2p(1-p)$, in which case the probability of a correct majority is the same as it was when there were $2n-1$ voters. With probability p^2 the last two voters are both for the better outcome, in which case only $n-1$ of the first $2n-1$ voters needed. Finally, if the last two votes are "wrong" there needs to be one more that a majority from the first $2n-1$ in order for the larger group to have a majority. We have

$$P(n-1;2n-1) = P(n;2n-1) + \binom{2n-1}{n-1} p^{n-1}(1-p)^n$$

and

$$P(n+1;2n-1) = P(n;2n-1) - \binom{2n-1}{n} p^{n}(1-p)^{n-1}$$

and so

$$\begin{aligned} P(n+1;2n+1) - P(n;2n-1) &= \\ p^2 \binom{2n-1}{n-1} p^{n-1}(1-p)^n - (1-p)^2 \binom{2n-1}{n} p^{n}(1-p)^{n-1} &= \\ \binom{2n-1}{n-1} [p(1-p)]^n (2p-1) > 0, & \end{aligned}$$

where the first equation uses $\binom{2n-1}{n-1} = \binom{2n-1}{n}$ and the inequality uses $p > .5$.

The second part of the theorem follows from the law of large numbers.

Since asymptotically the fraction of signals will be either very close to p or $1 - p$, any group decision rule that requires a fraction of votes between $1 - p$ and p to convict will implement the correct decision.

INFORMATIVE VOTING IS NOT STRATEGIC

For convenience, assume common q.
Given a signal of 0, a voter's posterior that $\pi = 0$ is:

$$\frac{p\pi}{p\pi + (1-p)(1-\pi)},$$

so a sincere voter will vote “guilty" after a signal of 0 if

$$\frac{p}{1-p}\frac{1-q}{q} > \frac{1-\pi}{\pi}.$$

Now imagine that k^* is the smallest value of k that satisfies:

$$\left(\frac{p}{1-p}\right)^{2(k^*+1)-n}\left(\frac{1-q}{q}\right) > \frac{1-\pi}{\pi} > \left(\frac{p}{1-p}\right)^{2k^*-n}\left(\frac{1-q}{q}\right)$$

k^* is well defined provided that you rule out boundary cases (eg, the left hand inequality holds when $k = 0$ or the right hand inequality holds when $k = n$) and ties (equations).
A simple argument shows that sincere voting is informative if and only if: $k^* = (n-1)/2$. (This choice of k^* reduces the exponent on the left-hand side to 1.)
Note that if k^* votes are needed to convict, then the inequality determines when a strategic voter will be informative.

QUESTION

What if different agents have different q?

CONCLUSIONS

1. Sincerity is informative only under majority rule.
2. Informative is rational only under k^* rule.
3. Rational voting will be both sincere and informative only when majority rule is rational and optimal.

SO WHAT?

1. When q are identical, then there should be no problem getting to efficiency.
   See McLennan, but the idea is to use k^* and sacrifice sincerity for information.
2. When q are identical, then why not share information? See Coughlan, but this is obvious. All agents report their private information, then take best action given pooled information. Honesty is an equilibrium.
3. What about q different?
   If they are not very different and N is large, then there are still no problems.

UNANIMITY WORKS BADLY

Does requiring unanimous votes for action 0 (now interpreted as “convict") avoid convicting the innocent? Not when voters are strategic.
FP show that if $N - 1$ guilty signals are enough to convince the jury to convict, then unanimity may be a bad ideas with strategic voters. The idea is now familiar. A strategic voter conditioning on the fact that other jurors want to convict will ignore his private information. For large N intermediate rules will converge to optimality while unanimity always involves convicting some innocents.

CRITICISMS

1. Unanimity is unreasonable for N large.
2. Why not share information?
3. No hung juries.