

Econ 208, Fall 2007: Problem Set 4: Possible Answers

Comments: Check the answer to Question 1 (c). Some perhaps should look at remarks about 2.

1. The point here was to show that if the hypotheses of the subgame-perfect folk theorem fail, then the conclusion may fail as well.

It is clear that the minmax value is 0 for all players. A somewhat tedious, but routine, computation establishes the second part. If $\sigma_j(1)$ is the probability that player j attaches to the first action, then there must be a player i with the property that either both of the others have $\sigma_j(1) \geq .5$ or both of the others have $\sigma_j(1) < .5$. In the first case j gets at least one quarter by playing the first strategy. In the second case j gets at least one quarter by playing the second strategy.

The third part was harder than people realized. Let v^* be the lowest subgame equilibrium payoff when the discount factor is δ (if you are worried about existence, let it be the inf). This equilibrium gives payoffs $(1 - \delta)u_i + \delta w_i^*$ to player i . Let player j be a player who can deviate in the first period and earn a payoff of at least one quarter. That means that this player's equilibrium payoff v^* must be at least

$$(1 - \delta)\frac{1}{4} + \delta v^*$$

because $w_i^* \geq v^*$ by definition. Hence $v^* \geq (1 - \delta)\frac{1}{4} + \delta v^*$ and so $v^* \geq 1/4$.

Consequently, the folk theorem's conclusion does not hold: the set of subgame perfect equilibrium payoffs is a strict subset set of individually rational feasible payoffs for all δ : payoffs can be no less than 1/4. It is clear that the assumptions of the theorem fail. The problem is that in order to punish a deviator the other players must suffer **and** it is impossible to reward those who punish a deviator without rewarding the deviator as well. Here is where the folk theorem in Nash equilibrium is easier. In that case, one can support low average payoffs with incredible threats.

2. Consider:

	Left	Right
Top	4,4	-100,5
Bottom	5,-100	0,0

If players play Tit-for-Tat, then the equilibrium path generates payoff 4. The best one-shot deviation leads to payoff of 5 today follow by alternating -1 and 5. This averages to $(1 - \delta)4 + \delta(-100 + 5\delta)/(1 + \delta)$ which is less than 4 if $\delta = .1$. So, tit-for-tat is immune from a one-shot deviation. On the other hand, deviating to a strategy that always cheats yields pay $(1 - \delta)5 > 4$ (when $\delta = .1$). There are many similar examples.

3. The basic idea is that when column is a short-run player, then she must best respond to beliefs. Without history, this means repeating one-shot Nash (part a). With history you can do better. In parts b and c, if the column player plays left as long as the row player has always played bottom (or, in (c) played $a = .5$) in the past) and otherwise plays right, then it is a best response for the row player to always play bottom on the equilibrium both and top off the path. This generates a nice payoff.

In general, the long-run player can obtain his "leader's" payoff in each stage game when δ is close to one: $\max u_1(s_1, BR(s_1))$, where $BR(s_1)$ is column player's best response to s_1 (in the stage game). Note that this might be lower than the largest individually rational and feasible payoff for the row player.

4. As I wrote this problem, “doubling” the game does not change the strategic possibilities. I intended to construct an example in which adding an extra market made it easier to cooperate because it enabled larger punishments relative to the rewards. It is possible to do this in interesting settings. If you are interested, look at a paper by Bernheim and Whinston on Multimarket Contact and Collusive Behavior, 1990 *Rand Journal of Economics*.
5. The various parts of this problem just changed the punishments available. When you are playing subgame perfect equilibrium and restrict attention to symmetric strategies, then the last period payoff must be $8/9$ on the equilibrium path. This is the worst equilibrium payoff, hence players must play an equilibrium in the first period (because there is no ability to punish). Again, restricting to symmetry (because top and left are dominated by a mixture of the other two strategies), players can only repeat the stage-game mixed equilibrium.

When Nash is the solution concept, you can use a non-equilibrium punishment after a first-period deviation. This punishment can be asymmetric because the history triggering the punishment will distinguish the players. Hence a player will lose at most $58/9$ from a deviation. This is not enough to support non equilibrium behavior in the first period (although some of you pointed out that it would be enough if one could observe mixed strategies).

Finally, when correlation is permitted, then it is possible to have a symmetric correlated equilibrium yielding expected payoffs of 4.5 in the final stage. Hence the punishment available in Nash equilibrium goes up to 9.5 and in subgame perfect equilibrium it goes up to $4.5 - 8/9$. These are still not enough to support a top left outcome in the first period. Several of you correctly observed that if it were possible to observe mixed strategies, then one could obtain higher payoffs in the first period. Otherwise, you would be able to do more if the attractive payoff, 20, was a bit smaller.