Econ 208, Fall 2007: Problem Set 3: Possible Answers

Comments: For the most part the problems were straightforward (or, at least, solutions were available in texts), so I will not provide detailed solutions. Please note that I asked you to prove something that was not true in the tenth question. This was deliberate (you have a right to be skeptical). Every two or three years I try this to give students a taste of what research is like. Once you have trouble proving something, you should (a) figure out what you can prove; (b) try to find a counter example; and (c) try to prove the conclusion of the theorem under stronger assumptions. In this case, all three of these things are possible (you can decide whether they are hard). I realize that as students you are conditioned to answer the question as it is asked, but as researchers you will necessarily need to modify your "problems" until you can solve them.

- 0. Easy. (A few people simple restated the identities rather than interpreting them.)
- 1. Straightforward.
- 2. It is useful to note that if $W(s) = \wedge_{i \in G} P_i(s)$, then $W(s) = C_G(W(s))$. We know

  $$\mathcal{E}_G(W(s)) = \cap_{i \in G} K_i(W(s)) = \cap_{i \in G}\{s \in S : P_i(s) \subset W(s)\} = W(s).$$

  The first equation is the definition of E_G, the second equation is the definition of K_i, and the third is the definition of meet. The claim follows by induction.

  If $W(s) \subset E$, we show that $s \in C_G(E)$. By monotonicity of K_i, we have that $K_i(W(s)) \subset K_i(E)$. Taking intersections we have that $\mathcal{E}_G(W(s)) \subset \mathcal{E}_G(E)$. By induction (and monotonicity of K_i) it follows that $W(s) \subset C_G(E)$, which implies that $s \in C_G(E)$.

  Conversely, if $s \in C_G(E)$ then $s \in K_i(E)$ for all i, so by Problem 1, $P_i(s) \subset K_i(E)$ for all i and therefore since $K_i(E) \subset E$, it follows that $P_i(s) \subset E$ for all i, which implies the result by the definition of the meet.
- 3. Straightforward.
- 4. Straightforward.
- 5. Straightforward
- 6. Straightforward
- 7. Straightforward
- 8. Straightforward
- 9. Straightforward

  If 4-6 are not straightforward, then it least it appeared that you could find the solutions. A couple of people seemed confused by the distinction between self evident and common knowledge. Here are some ideas: A self-evident event must be common knowledge when the event occurs. Any super set of a set that is common knowledge is common knowledge when the smaller event occurs. On the other hand, a super set of a self evident set is not necessarily self evident. If there are two people and one has the finest partition while the other can only distinguish the first point in S from the rest of the rest (S has more than two elements), then the first point $s^* \in S$ is self evident, but any larger proper subset of S that contains s^* is common knowledge at s^*.

- 10.

  The idea of this question is that you could use K as a primitive with which to generate partitions.

  The definition says, roughly, the partition element containing s is the smallest event that you know has occurred when s occurs. $K(S) = S$ guarantees that you always know when "something" happens.

  First note that $P(s)$ is well defined because S contains s and so it is defined to be the intersection of a finite (non-empty) collect of sets. Next note that $s \in P(s)$ for all s. This is because $K(E) \subset E$ for all E, hence if $s \in K(E)$, then $s \in K$. Hence $P(s)$ is the intersection of sets E that all contain s. Consequently $P(s)$ contains s and therefore the union of $P(s)$ is S.

  Consider the following K function defined on $S = \{1,2,3\}$ where $K(S) = S$, $K(\{1,2\}) = \{1,2\}, K(\{1,3\}) = \{1,3\}, K(\{2,3\}) = \{3\}, K(\{1\}) = \{1\}, K(\{2\}) = \phi, K(\{3\}) = \{3\}$. You can check that K satisfies all of the properties of Problem 0. Note that $P(1) = \{1\}, P(2) = \{1,2\}$, and $P(3) = \{3\}$ is not a partition. Hence the conditions given in the problem were not sufficient to guarantee that P is a partition.

  Since $K(\cap E_i)) = \cap K(E_i)$ and $P(s)$ is the intersection of events E_i where $s \in \cap K(E_i)$ it must be that $s \in K(P(s))$. Hence, without further assumptions we can show that $K(P(s)) = P(s)$. Suppose that $t \in P(s)$, then $t \in K(P(s))$ so $P(t) \subset P(s)$. This means that while the sets $P(\cdot)$ are not disjoint, they are nested.

  Suppose in addition that

  $$\neg K(E) \subset K(\neg K(E))$$

  Since $K(E) \subset E$, this implies that $\neg K(E) = K(\neg K(E))$. This condition says that you know if you do not know E, then you know that you do not know E. You can confirm, using the first problem, that this condition holds if the knowledge operator is defined from a given partition. Conversely, suppose that $t \notin P(s)$. Since $P(s) = K(P(s))$ it follows that $t \notin K(\neg P(s))$ so, by the definition of $P(t)$, $P(t) \subset \neg P(s)$.

  Combining what we have learned we have that if $t \in P(s)$ then $P(t) \subset P(s)$, while if $s \notin P(t)$, then $P(s) \cap P(t) = \phi$. This means that $P(t)$ and $P(s)$ are either identical or disjoint.

- 11. Agent i's information partition is $\{(x_i, 0), (x_i, 1)\}$. Given that at the given state it is common knowledge that $x \neq 0$ we can treat this point as if it is not in the state space. This means that i can infer that his face is muddy iff $x_{-i} = 0$. Otherwise he does not know whether his face is muddy. After k rounds of silence, one rules out states in which any agent sees $k-1$ dirty faces. Hence, by induction, if there has been silence for $k-1$ stages then an agent who sees exactly $k-1$ muddy faces knows that his face must be muddy too.