**Econ 208, Fall 2007: Problem Set 3**

**Instructions:** Due: November 26, 2007

This is a short course in common knowledge. You can find most of the information in the texts, but I encourage (urge) you to work out the problems on your own. The general structure is abstract, but the specific exercises, with no more than two exceptions, are straightforward applications of definitions. As you do the exercises, make an effort to think about what the concepts mean and their (potential) relevance to game theory.

Let $S$ be an abstract set of states. An event $E$ is a subset of $S$. An event holds at state $s \in S$ if and only if $s \in E$. An agent $i$ has a partition $\mathcal{P}_i$ of $S$ (a partition is a collection of subsets of $S$ that are pairwise disjoint and whose union is $S$). Denote by $\mathcal{P}_i(s)$ the element of the partition $\mathcal{P}$ that contains $s$.

A knowledge operator for player $i$ is a mapping $K_i : 2^S \to 2^S$ such that

$$K_i(E) = \{s \in S : \mathcal{P}_i(s) \subset E\}.$$

$K_i(E)$ is the event in which $i$ knows $E$. ($2^S$ is the set of all subsets of $S$.)

**Problem 0.** If $K$ is a knowledge operator, then show $K(S) = S$, $E \subset F$ implies $K(E) \subset K(F)$, $K(E) \cap K(F) = K(E \cap F)$, and $K(E) \subset E$. Interpret these identities.

Problem 0 should be easy. Do not go further until you can do it.

**Problem 1.** Show that $K_i(E) = \cup_{P_i \subset E} P_i$.

If $G$ is a set of agents, then define $\mathcal{E}_G : 2^S \to 2^S$ to the function:

$$\mathcal{E}_G(E) = \cap_{i \in G} K_i(E).$$

$E_G(E)$ is the event that everyone in $G$ knows $E$.

Common knowledge of an event $E$ among agents in $G$, $C_G(E)$, is the event of all the players knowing $E$, all knowing that they all know $E$, and so on. If we define $\mathcal{E}_G^n(E)$ inductively by $\mathcal{E}_G^1(E) = \mathcal{E}_G(E)$ and $\mathcal{E}_G^{n+1}(E) = \mathcal{E}_G(\mathcal{E}_G^n(E))$ for $n > 1$, then

$$C_G(E) = \cap_{k=1}^{\infty} \mathcal{E}_G^k(E).$$

The meet of two partitions $\mathcal{P}$ and $\mathcal{P}'$ is the finest partition coarser than both $\mathcal{P}$ and $\mathcal{P}'$. We denote the meet $\mathcal{P} \wedge \mathcal{P}'$. The joint of the two partitions, denoted $\mathcal{P} \vee \mathcal{P}'$ is the coarsest partition finer than both $\mathcal{P}$ and $\mathcal{P}'$. The element of the join containing $s$ is the intersection of $\mathcal{P}_i(s)$ and $\mathcal{P}_i'(s)$.

**Problem 2.** Show that $s \in C_G(E)$ if and only if $(\vee_{i \in G} \mathcal{P}_i)(E) \subset E$.

An evident $E$ is self-evident between 1 and 2 if for all $s \in E$, $\mathcal{P}_i(s) \subset E$ for $i = 1$ and 2.

**Problem 3.** Suppose that $S = \{s_1, \ldots, s_7\}$ let $\mathcal{P}_i$ be partitions for player $i$, $i = 1$ or 2, and let $K_i$ be the associated knowledge functions. If

$$\mathcal{P}_1 = \{\{s_1, s_2\}, \{s_3, s_4, s_5\}, \{s_6, s_7\}\}$$

and

$$\mathcal{P}_2 = \{\{s_1\}, \{s_2, s_3, s_4, s_7\}, \{s_5\}, \{s_6\}\},$$

Does there exist a non-trivial event that is self evident between the two agents? Are any events common knowledge?

**Problem 4.** Show that $E$ is self-evident between agents 1 and 2 if and only if $K_i(E) = E$ for $i = 1$ and 2. Further show that $E$ is self-evident if and only if it is equal to a union of elements of $\mathcal{P}_i$ for $i = 1$ and 2.

**Problem 5.** Show that an event $E$ is common knowledge in the state $s$ if and only if there is a self-evident event $F$ such that $s \in F \subset E$.

The first big result in the theory of knowledge is the "Agreement Theorem" due to Aumann. The question is: Can it be common knowledge between two agents with the same prior belief that agent 1 and agent 2 assign different probabilities to the same event? The answer is no. This result is important for the foundation of games with incomplete information because there the standard assumption is that players have identical prior beliefs. If (following the constructions about hierarchies of beliefs) we assume that it is common knowledge that the players have the same prior, whenever their beliefs are common knowledge, the beliefs must agree.

Let $\rho$ be a probability measure on $S$ (the prior) and let $\mathcal{P}_i$ be player $i$'s partition. If $E$ is an event and $\eta_i$ is the conditional probability of $E$ given $P_i(s)$ (written $\rho(E \mid \mathcal{P}_i(s))$), then $i$ assigns probability $\eta_i$ to the event $E$. We can construct an event $F$ that is interpreted as the event in which $i$ assigns probability $\eta_i$ to $E$: $F = \{s : \rho(E \mid \mathcal{P}_i(s) = \eta_i\}$.

Now for the theorem:

Suppose that $S$ is finite and agents 1 and 2 share a common prior $\rho$. If it is common knowledge between the agents that in state $s^* \in S$ that individual $i$ assigns probability $\eta_i$ to $E$, then $\eta_1 = \eta_2$.

**Problem 6.** Under the assumptions of the theorem, show that there exists a self-evident event $F$ containing $s^*$ in which both agents assign probability $\eta_i$ to $E$.

**Problem 7.** Use the previous problems (4-6) to show that $F$ is Problem 6 is a union of elements of $\mathcal{P}_i$ for both $i$.

**Problem 8.** Show that for any nonempty disjoint sets $C$, $D$, and $E$ of a finite set, if $\rho(E \mid C) = \rho(E \mid D) = a$, then $\rho(E \mid C \cup D) = a$.

**Problem 9.** Deduce the theorem.

Instead of taking the partition structure as primitive, one can start with a knowledge function $K$ that satisfies the properties in Problem 0.

**Problem 10.** Given a knowledge function $K$ satisfying the properties in Problem 0, show that the family of sets $\mathcal{P}(s) = \cap\{E \subset S : s \in K(E)\}$ is a partition of $S$. Would this still be true if $K$ failed to satisfy some of the properties in Problem 0? Which ones?

Knowledge operators are a formal device for handling a variety of logic puzzles. You can read about the "coordinated attack problem" and the "puzzle of the hats" and the "electronic mail game." Here is a related story.

$n$ children are playing together. After some exciting outdoor play exactly $k$ of them get mud on their foreheads. Each can see the mud on others, but no child can see his or her forehead. A man walks by and says: "At least one of you has mud on your forehead." As long as $k > 1$, all of the children knew this already. Now the man asks the following question: "Does any of you know whether you have mud on your forehead?" He keeps asking this question until the children respond. Assuming that the children are all trained in logic and they answer the question honestly, what happens?

**Problem 11.** Describe a state of the world as a list of $n$ binary numbers $x_i$, where $x_i$ is 1 if child $i$ is muddy and zero otherwise. Describe the information partitions of the children and how they change with each statement by the man and with each response by the children.