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Economics 208: Problem Set II, Possible Answers

Most of you did fine on the first and sixth questions (although there was limited imagination on the choice of examples). My answer to 6 is different from and, I think, more instructive than the modal student response. Question 2 was vague, but no one did more than restate definitions. Please reflect on this: both Nash and trembling hand perfect impose some continuity conditions. On the surface, at least to me, the conditions are inconsistent. One resolves the inconsistency by noting that they require different topologies and different senses of continuity of set-valued mappings. Answers to Questions 3 and 4 were fine, for the most part. On Question 3, there is a unique robust prediction – the first player sets $d = 0$ are earns 5, although no single strategy profile is robust to all perturbations. The answers to Question 5 were weak. Most people stated (or acted as if) the plaintiff had to treat all offers as if they were from a randomly selected type. This is certainly not true.

1. Let G_n be a sequence of games and σ_n a sequence of equilibrium strategy profiles. Since the range of the equilibrium correspondence is compact, uhc is equivalent to closed graph. By compactness, a limit point of the sequence of strategy profiles exists. By continuity of payoffs any limit inherits the best response property.

   Consider:

	s_j^1	s_j^2
s_i^1	$1, 1$	$0, 0$
s_i^2	$0, 0$	a, a

   When $a = 0$ the set of Trembling Hand Perfect equilibria is a singleton $((s_i^1, s_j^1))$. When a is positive, there are two THPE. Hence the limit of (s_i^2, s_j^2) is not an equilibria and the correspondence is not uhc.

   When $a < 0$, the game above has a unique Nash equilibrium, hence since the equilibrium set explodes at $a = 0$ the correspondence is not lower hemi continuous.

2. There are two differences: what is the appropriate notion of continuity (upper or lower hemicontinuity) and what is the appropriate topology (which games are "close"). In the previous problem, all perturbations of payoffs were allowed. This means that many games are close. The requirement that any equilibrium of a close game must be close to an equilibrium of a given game therefore puts a lot of pressure on the solution concept to be big. In the definition of THPE, fewer games are close (because nearby games come from strategy perturbations not all payoff perturbations). Further, in the definition of THPE we require that any equilibrium must be the limit of equilibria from some nearby games. This condition is more like lower hemi continuity than upper hemi continuity (but is weaker than lhc because it only requires an approximation from some nearby sequence rather than all of them).

3.

	11	12	21	22
$n1$	$5, 1$	$5, 1$	$0, 0$	$0, 0$
$n2$	$0, 0$	$0, 0$	$1, 5$	$1, 5$
$y1$	$3, 1$	$-2, 0$	$3, 1$	$-2, 0$
$y2$	$-2, 0$	$-1, 5$	$-2, 0$	$-1, 5$

   The column strategies are of the form ab where a is the response to $d = 0$ and b is the response to $d = 2$. The row strategies are of the form ab where $a = n$ if $d = 0$ and $a = y$ if $d = 2$ and b is what the player does subsequently. (Officially, this player's strategy should also specify an action for the "foregone" game, but I ignore the redundancies.) Examination of the payoff matrix reveals that the bottom strategy is strictly dominated by the top strategy. After it is deleted, column strategies 12 and 22 are weakly dominated (by

11 and 21 respectively). Hence the row player will play $n1$ and the column player will play 11. While this is the unique outcome that survives iterative deletion of dominated strategies in the natural order, it is possible to construct trembles that make 11 perform badly relative to 12 so, strictly speaking there is no strategy immune from all perturbations.

The bottom strategy will not be used in any Nash Equilibrium. $(n1, 11)$, $(n1, 12)$, $(n2, 22)$, and $(y1, 21)$ are pure-strategy equilibria. There are mixed equilibria in which row plays $n1$ and column mixes between 11 and 12, row plays $n2$ and column mixes between 21 and 22 (but plays 21 with probability less than .8), row plays $n1$ and column randomizes between 11 (probability less than .6) and 21), and in which row plays $n1$ with probability 5/6 and $n2$ with probability 1/6 and column randomizes between her strategies to leave row indifferent.

The curious thing about this example is that by adding an option for player 1 (the ability to set $d = 2$) you can modify the game in player 1's favor even though (a) player 1 does not use the option in equilibrium and (b) the option (to throw away payoffs uniformly) does not look attractive. Game theory is strange sometimes.

4. Suppose that there are 4 players and each bids 0 with probability p. We need each player to be indifferent. Consider the problem of one of the players:

Sum of Others	prob	Bid 0	Bid 100
0	p^3	.25	1
100	$3p^2(1-p)$	0	1
200	$3p(1-p)^2$	0	.25
300	$(1-p)^3$	.25	.25

Plainly 100 is a best response, so you cannot have a nondegenerate mixed equilibrium.

Sum of Others	prob	Bid 0	Bid 100
0	p^4	.20	1
100	$4p^3(1-p)$	0	.5
200	$6p^2(1-p)^2$	0	$\frac{1}{3}$
300	$3p(1-p)^3$	0	0
400	$(1-p)^4$	1	.2

In this case there is an equation to solve. I think that the equation is

$$.2p^4 + (1-p)^4 = p^4 + 2p^3(1-p) + 2p^2(1-p)^2 + .2(1-p)^4,$$

which appears to reduce to a cubic with a unique solution in $(0, 1)$ $(p = 1/3)$.

5. In one class of equilibria, both types of defendant make a common offer. If the common offer is s^*, then s^* must be accepted if it is greater than $H/2$ and must be rejected if it is less. You can support on equilibrium outcome with $s^* > 0$ provided that defections are not attractive. Lower offers will be attractive if they are accepted with probability one. A lower offer will be attractive if it is rejected and $s^* > c + t$. Since a lower offer must be unattractive to both types, this means that $s^* \leq c$. Any pooled offer $s^* \in [H/, 2, c]$ can be supported as an equilibrium. (A pure strategy response for the plaintiff that "supports" this outcome is to accept any offer greater than s^* and reject any lower one. In fact, higher offers can be rejected with any probability and lower offers can be accepted with positive probability as long as the $t = 0$ defendant does not want to deviate. Of course, there are no equilibria of this form if $c < H/2$.

   If the common offer is less than $H/2$, then it must be rejected. This can be part of a Nash equilibrium if, for example, the plaintiff rejects all offers. On the other hand, it would be silly (weakly dominated) to reject an offer greater than $H + c$. (Accepting such offers does not expand the set of equilibrium outcomes.)

   So there are a large number of pooling equilibrium outcomes and each of these outcomes is actually induced by a large number of equilibria.

Another class of equilibria are separating equilibria. In a separating equilibrium, different types make different offers. Suppose that the offer of the $t = 0$ type is accepted with probability one. Call the offer s_0. Suppose that the $t = H$ type offers s_H, which is accepted with probability p_H. In order for this to be a separating equilibrium it must be that $s_0 \geq p_H s_H - (1 - p_H)c$ ($t = 0$ prefers s_0) and $s_0 \leq p_H s_H - (1 - p_H)c - (1 - p_H)H$ ($t = H$ prefers s_H). This is impossible. So s_0 must get rejected. If it is always rejected, then either s_H is always rejected or $c \leq s_H$. So you can have equilibria in which $t = 0$ makes an offer than is rejected and $t = H$ makes an offer than is at least H (so it is accepted) and greater than c (so that it does not attract $t = 0$). Outcomes like this are equilibria (you can assume, for example, that all other offers are rejected). Finally, the offer $t = 0$ could be accepted with positive probability. In a separating equilibrium, this means that the plaintiff is indifferent between accepting (and getting s_0) or rejecting and getting 0 (because t is known). Hence $s_0 = 0$. If this offer is accepted with probability p_0, then H pays $(1 - p_0)(c + H)$ As long as $p_0 < 1$, this means that s_H must be accepted with positive probability and hence must be at least H. Since it is senseless (weakly dominated) for the plaintiff to reject offers higher than H, the only sensible possibility is for $t = H$ to offer H. There are many such equilibria (because these offers can be rejected with positive probability). It is reasonable to assume that such an offer would be accepted with probability one (because higher offers should be accepted). This creates an interesting candidate equilibrium in which $s_H = H$, $s_0 = 0$, intermediate offers are rejected (at least with sufficiently high probability), s_H is accepted and p_0 is such that $(1 - p_0)(c + H) \leq H$ or $c/(c + H) \leq p_0$.

Anything else? It is possible that with positive probability both defendant types play the same strategy. It is possible to construct these things.

What about robustness? If both types of defendant use the same offer with positive probability, then it is sensible to accept slightly lower offers (informally, to treat them as coming from the $t = 0$ type). If you have a perturbation in which it is much more likely for the $t = 0$ sender to make the lower offer, then equilibria of the perturbed game will not be near the equilibrium with the pool. Hence you are essentially left with the separating equilibrium in which $s_0 = 0$. An arguably more subtle argument suggests that the best prediction is $p_0 = c/(c + H)$. Roughly speaking, if H is not indifferent between the two offers made with positive probability, then a slighly positive offer should be interpreted as an offer from the $t = 0$ type and hence be accepted. There are published papers about this model. Let me know if you are interested.

6. Nothing in the problem says that i is a strategic player. Imagine a game in which player i is the third player and the other two players play:

	s_j^1	s_j^2
s_i^1	$5, 1, a$	$0, 0, b$
s_i^2	$4, 4, b$	$1, 5, a$

The Nash equilibria yield payoffs to player 3 of either a or $(17a + 8b)/25$. The correlated equilibrium that avoids the upper right corner and player the other three with equal probability yields $(a+2b)/3$. This payoff can be greater than both (if $a < b$) or less than both (if $a > b$).