

### Economics 208: Problem Set I, Possible Answers

Generally the answers were sensible. Many people failed to supply all details. This leads to problems because these people also made claims, without justification, that were incorrect. Specific Comments:

1. Most people concluded that since 200 was not a best response to a pure strategy, then it cannot be a best response to any beliefs. This is not necessarily true. You must check that it is not a best response to an arbitrary (mixed) strategy.
2. Several people said incorrect things about the nature of mixed strategies in this problem (and elsewhere). In the second assignment, I ask you to reconsider the problem of describing mixed strategy equilibria.
3. Answers were generally fine.
4. Again, answers were fine, but some did not have the patience (or had the good sense not) to write down a complete description of equilibria.
5. Please make sure that you understand the argument.

1. Obviously 200 is not a best response to any pure strategy. This is not enough to conclude that 200 is strictly dominated (or not rationalizable). A more complete answer is that if your opponent bids 100, your strict best response is 100. Otherwise, if  $n$  is the highest bid made by your opponent with positive probability, then you do better bidding  $n - 1$  than by bidding 200. (The payoff from the two strategies is the same when your opponent bids less than  $n - 1$ , but bidding  $n - 1$  does strictly better than 200 when opponent bids either  $n - 1$  or  $n$ .) Consequently, 200 is not a best response to any strategy. Consequently it is not rationalizable. By symmetry, opponent won't bid 200. If you iterate, then you arrive at a unique rationalizable strategy profile (100, 100). (In experiments people do not play this way!)
2. While it is true that players always have a best response in the set  $\{0, 100\}$ , no strategies are strictly dominated. For example, any  $x > 0$  is a best response when all other players select 0.

First suppose that all players play pure extreme (0 or 100) strategies. If  $1 < k < N - 1$  agents bid 100, then the target is  $2k/3N$ . If this is equal to .5, then everyone wins and there is no incentive to deviate (deviation makes your payoff zero). You can have an equilibrium with  $2k/3N < 50$  (so that the  $k$  high bidders win) if it does not pay to switch from 0 to 100:  $2(k + 1)/3N > 1/2$  and  $1 < k < N - 1$ . Similarly, you can have an equilibrium with  $2k/3N > 1/2$  provided that it does not pay to switch from 100 to 0. Putting these pieces together,  $k$  is an equilibrium if  $k \in (3N/4 - 1, 3N/4 + 1)$  (this interval always contains an integer) and  $N > 3$  (to avoid boundary cases). The lower-order cases need separate arguments (but you can check that all playing 100 is an equilibrium when  $N = 2$  or 3).

It is possible to have equilibria in which players make intermediate bids. Strategies must be arranged so that no one can win by deviating. I will not provide a systematic discussion, but note that if one player bids 0 and another bids 100, then you will have an equilibrium provided that all other bids are strictly between 0 and 100 and two-thirds of the average is equal to 50. In this way, the extreme guys split the prize, but any movement to the extremes will shift the winner to the other extreme.

3. (a) First note that all players must use the same strategy in any pure-strategy equilibrium. If not, then at least one player is not bidding the average. Further, there must be a player whose payoff is less than one third. (If all payoffs are one third, then all guesses are equally close to the average. If guesses are not all the same, then one guess must be below the average and another above it. If these guesses are equidistant to the average, then the third bid must be exactly the average.) This player can improve his payoff (to at least one third) by bidding the average of the other two players' bids.

It should be clear that any strategy profile in which the three players make the same bid is a Nash equilibrium. (They each get  $1/3$  for following the equilibrium strategy and get zero from a deviation. This time I'll supply symbols. If the common bid is  $s$ , then by bidding  $b > s$  instead, the distance to the average is  $2(b - s)/3$  for the deviant and  $(b - s)/3$  for the other two. Similarly for  $b < s$ .

- (b) The pure strategy equilibria of the first part remain. Indeed, if two players play the same pure strategy, then the only possible equilibrium is for the third player to also play that strategy.

It cannot be an equilibrium for two players to play different pure strategies. For example, if player 1 bid 0, player 2 bid 1, and player 3 bid 0 with probability  $p \in (0, 1)$ , then player 2 does better by bidding 0 (bidding 0 wins with probability  $p/3 + (1 - p)/2$  bidding 1 wins with probability  $(1 - p)/2$ ). So in a non-degenerate mixed-strategy equilibrium, at most one player plays a pure strategy. You can check that if the first player plays a pure strategy and the second player uses a non-degenerate mixture, then the best response of the third player is to do the same thing as the first player. This is a contradiction, so either all players play the same pure strategy or all use a non-degenerate mixture. If player  $i$  plays 0 with probability  $p_i \in (0, 1)$ , then the payoff to the pure strategy 0 is equal to the payoff to 1. For example, for player 3 it must be the case that the expected payoff from playing zero:

$$\frac{p_1 p_2}{3} + \frac{p_1(1 - p_2)}{2} + \frac{(1 - p_1)p_2}{2} + (1 - p_1)(1 - p_2)$$

is equal to the payoff from playing one:

$$p_1 p_2 + \frac{p_1(1 - p_2)}{2} + \frac{(1 - p_1)p_2}{2} + \frac{(1 - p_1)(1 - p_2)}{3}.$$

Hence you need:  $p_1 p_2 = (1 - p_1)(1 - p_2)$ . By symmetry, you need:  $p_i p_j = (1 - p_i)(1 - p_j)$  for all  $i \neq j$ . Hence  $p_i$  is independent of  $i$  and  $p_i^2 = (1 - p_i)^2$  so that  $p_i = .5$ . It is straightforward to check that this mixture probability indeed generates a Nash Equilibrium.

4. (a) Anyone can gain by deviating, so playing the same thing cannot be an equilibrium.  
 (b) Suppose that the players' strategies are  $a$ ,  $a$ , and  $b$ , for  $b > a$ . It follows that  $b$  wins. This will be an equilibrium if neither player selecting  $a$  can gain from deviating. If a deviation is attractive, then a deviation to either 0 or 1 will be attractive. Hence we need it to be the case that if one of the  $a$  strategies moves to zero,  $b$  is still further from the mean:

$$\frac{a + b}{3} < b - \frac{a + b}{3}$$

and if one of the  $a$  strategies moves to one, then  $a$  is now furthest from the mean (the middle strategy will never be closest):

$$1 - \frac{1 + a + b}{3} < \frac{1 + a + b}{3} - a.$$

The first inequality simplifies to  $b > 2a$  while the second requires  $b > (1 + a)/2$ . Hence equilibrium requires  $b > \max\{2a, (1 + a)/2\}$ .

You can derive a similar condition when two players make the higher bid.

- (c) Suppose the bids are  $a < b < c$ .  $b$  cannot win. If both  $a$  and  $c$  win, then both must be boundary strategies (otherwise, it would pay for one to move to the boundary). There is an equilibrium in which  $a = 0$ ,  $c = 1$  and  $b = .5$  ( $b$  cannot gain by moving and by locating at the midpoint the other two strategies are equidistant from the average.) Assume instead that only one of the extreme strategies wins. For concreteness, assume that it is  $c$ . In order to be an equilibrium, it cannot be profitable for  $a$  to deviate to zero (you can check that if the player using  $b$  has a profitable deviation to zero, then the same deviation will be profitable for the player using  $a$ ) and it cannot be profitable for the player using  $b$  to deviate to one. Hence equilibrium requires  $c > \max\{2b, (1 + 3b - 2a)/2\}$ . (There is also the possibility that  $a$  is the unique winner.)

(d) This really is in the text.