Econ 205 - Slides from Lecture 15

Joel Sobel

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Simple Allocation

$$\max x_1 \cdots x_n \text{ subject to } \sum_{i=1}^{k} x_i = K.$$

This problem has an equality constraint. If $v = x_1 \cdots x_n$, then we can write the first-order conditions as $v = \lambda x_i$ for all i. Hence $kv = \lambda K$ and therefore each $x_i = K/k$.

A Famous Inequality

Solve:

$$\max x_1^2 x_2^2 \cdots x_n^2 \text{ subject to } \sum_{i=1}^{n} x_i^2 = 1$$

is to set $x_n = (1/n)^n$ for all n.

Solution must satisfy

$$2\frac{f(x)}{x_i} = 2\lambda x_i,$$

which implies that the x_i are independent of i.

[Why is this a maximum and not a minimum?]

Given any n positive numbers $a_1, \ldots, a_n$, let

$$x_i = \frac{a_i^{1/2}}{(a_1 + \cdots + a_n)^{1/2}}, \text{ for } i = 1, \ldots, n.$$

It follows that $\sum_{i=1}^{n} x_i^2 = 1$ and so

$$\left(\frac{a_1 \cdots a_n}{(a_1 + \cdots + a_n)^n}\right)^{1/n} \le \frac{1}{n}$$

Hype

It is possible to use techniques from constrained optimization to deduce other important results (the triangle inequality, the optimality of least squares, . . .).

Cobb-Douglas

Canonical consumer theory example.
Cobb-Douglas Utility function:

$$f(x) = x_1^{a_1} \cdots x_n^{a_n},$$

where the coefficients are nonnegative and sum to one.
Consumer problem:

$$\max f(x) \text{ subject to } p \cdot x \leq w$$

where $p \geq 0$, $p \neq 0$ is the vector of prices and $w > 0$ is wealth.
Since the function f is increasing in its arguments, the budget constraint must hold as an equation.

The first-order conditions can be written

$$a_i f(x)/x_i = \lambda p_i$$

or

$$a_i f(x) = \lambda p_i x_i.$$

Summing both sides of this equation and using the fact that the a_i sum to one yields:

$$f(x) = \lambda p \cdot x = \lambda w.$$

It follows that

$$x_i = \frac{w a_i}{p_i} \text{ and } \lambda = \left(\frac{a_1}{p_1}\right)^{a_1} \cdots (a_n p_n)^{a_n}.$$

In Cobb-Douglas example, you have explicit formulas for solution and value function.
You can manipulate these to confirm generate properties (homogeneity and envelope theorem).

Integration

1. Inverse to differentiation.
2. Integration averages.
3. Integration computes areas.

The first property provides a method for solving differential equations.
The second property is useful for probability and statistics.
The third property is related to the second.

Integral as Generalized Average

Given a set of N numbers, you average them by adding the numbers up and then dividing by N. How do you generalize this to a situation in which you are given an infinite set of numbers to average?

Assume that $f : [a, b] \longrightarrow \mathbb{R}$.

- First guess: Pick some $x \in [a, b]$ and say that the average is equal to $f(x)$.
- Great if f is constant.
- Second guess: Find max and min of f. (Bounds for average.)
- Inspiration: Divide interval and repeat.

Setting Up

A *partition* P of $[a, b]$ is a finite set of points $x_0, x_1, \ldots, x_n$, where

$$a = x_0 \leq x_1 \leq \cdots \leq x_{n-1} \leq x_n \leq b$$

Given such a partition, define let $\Delta_k \equiv x_k - x_{k-1}$ and

$$L_f(P) = \sum_{k=1}^{n} m_k \Delta_k$$

$$\begin{aligned} m_k &\equiv \inf_{x \in [x_{k-1}, x_k]} f(x) \\ &= \inf \{f(x) \mid x \in [x_{k-1}, x_k]\} \end{aligned}$$

$$U_f(P) = \sum_{k=1}^{n} M_k \Delta_k$$

$$\begin{aligned} M_k &\equiv sup_{x \in [x_{k-1}, x_k]} f(x) \\ &= \sup \{f(x) \mid x \in [x_{k-1}, x_k]\} \end{aligned}$$

Properties

1. Well defined even if f is not continuous.
2. For each partition P, $L_f(P) \leq U_f(P)$.
3. Also, $L_f(P)$ is an underestimate of the "average."
4. $U(P)$ is an overestimate.

Continue

1. Subdivide the partition P by dividing each subset of P into two non-empty pieces.
2. This leads to a new partition P' and new values $L_f(P') \geq L_f(P)$ and $U_f(P') \leq U_f(P)$.
3. Reason for claim above:

$$\inf_{x \in [x_{k-1}, x_k]} f(x) \leq \inf_{x \in [x_{k-1}, z]} f(x) + \inf_{x \in [z, x_k]} f(x) \text{ for all } z \in [x_{k-1}, x_k]$$

4. Process generates an increasing sequence of lower estimates of the average value of f.
5. and a process that generates a decreasing sequence of upper estimates of the average value of f.
6. These sequences must converge.

Definition

Let $f : [a, b] \longrightarrow \mathbb{R}$ and suppose that P_r is a sequence of partitions

$$a = x_0(r) \leq x_1(r) \leq \cdots \leq x_{n-1}(r) \leq x_{n_r}(r) \leq b$$

such that $\Delta_k(r) = x_k(r) - x_{k-1}(r)$ goes to zero as r approaches infinity for all k. f is *integrable* if

$$\lim_{r\to\infty} L_f(P_r) \text{ exists and is equal to } \lim_{r\to\infty} U_f(P_r). \tag{1}$$

If f is integrable then we denote the common limit in (1) by $\int_a^b f(x)dx$.

Comments

1. Lots of ways to subdivide.
2. It won't matter (if upper and lower limits converge).
3. If a function is integrable, then it does not matter whether you evaluate f inside each partition element using the sup, the inf or any value in between.
4. If a function is integrable, then the integral (divided by $b - a$) is a generalization of average value.

What is Integrable?

- Not everything: Consider the function that is 1 on the rationals and 0 on the irrationals.
- Continuous functions are.
- Bounded Monotonic Functions are.

Generalizations

- The definition above is the definition of the *Riemann Integral*.
- A *Riemann-Stieltjes Integral* is a generalization of the Riemann Integral in which the "length" of the interval is not uniform. The contribution of a term in the sum is:

$$(\alpha(x_k) - \alpha(x_{k-1})) \inf_{x \in [x_{k-1}, x_k]} f(x)$$

  instead of

$$(x_k - x_{k-1}) \inf_{x \in [x_{k-1}, x_k]} f(x)$$

  for some strictly increasing function α.
- Stochastic calculus (used in finance) extends the class of integrable functions.

Measure Theory

- Measure theory defines integrals of functions defined on abstract sets equipped with a "measure" that gives you the "size" of various sets.
- One example of a measure is a probability distribution.
- The integrals used in measure theory are called expectations by probability theorists.
- Lebesgue Measure is the most common measure.
- Lebesgue measure defines the size of the interval $[a, b]$ to be $b - a$ and, more generally, the size of rectangles to be their (standard) area.
- Integral is defined as limit of average of functions that take on only finitely many values.
- The basic theory is similar to the theory of Riemann integration and you should not be intimidated when people talk about Lebesgue integrals.
- But the rational/irrational function is Lebesgue integrable.

Results

Theorem

If

$$f(x) = c, \quad \forall x \in [a, b]$$

Then

$$\int_a^b f(x)dx \quad = \quad c(b-a)$$

Definition

Take $f\colon (a, b) \longrightarrow \mathbb{R}$.
Suppose $\exists\, F\colon (a, b) \longrightarrow \mathbb{R}$, continuous and differentiable on (a, b).
If

$$F'(x) = f(x),$$

$\forall x \in (a, b)$.
Then F is called the *antiderivative* of f.

Example

$$f(x) = x^2$$

$$\implies F(x) = \frac{1}{3}x^3$$

also could have

$$F(x) = \frac{1}{3}x^3 + 6$$

Theorem

If F and G are both antiderivatives of f on $[a, b]$, then

$$G(x) = F(x) + c$$

Two antiderivatives differ by a constant.
You have seen this result before: It says that a function that has a derivative zero must be constant.

Fundamental Theorems of Calculus

Theorem (Fundamental Theorem of Calculus)

If

$$f : X \longrightarrow \mathbb{R}$$

is continuous and X is an open interval then, for $a \in X$ and F defined by

$$F(x) \equiv \int_a^x f(t)dt, \ \ \forall x \in X,$$

F is differentiable and

$$F'(x) = f(x), \ \ \forall x \in X$$

Interpretation

The theorem states that differentiation and integration are inverse operations in the sense that if you start with a function and integrate it, then you get a differentiable function and the derivative of that function is the function you started with.

Proof

Proof.
By the definition of the integral,

$$h \sup_{y\in[x,x+h]} f(y) \geq F(x+h)-F(x) = \int_x^{x+h} f(x)dx \geq h \inf_{y\in[x,x+h]} f(y)$$

the result follows from dividing through by h and taking limits. □

Part II

Theorem

This theorem is a converse of the first result.
If F is an antiderivative of

$$f : [a, b] \longrightarrow \mathbb{R}$$

then,

$$\begin{aligned} \int_a^b f(x)dx &= F(b) - F(a) \\ &= [F(x)]_a^b \end{aligned}$$

Proof.

By the mean value theorem $F(y) - F(x) = f(t)(y - x)$ for some t between x and y. It follows that for any partition P

$$U_f(P) \geq \sum_{k=1}^{n} (F(x_k) - F(x_{k-1})) \geq L_f(P) \tag{2}$$

because for each element of the partition $F(x_k) - F(x_{k-1}) = f(t_k)(x_k - x_{k-1})$ and $f(t_k) \in [\inf_{x \in [x_{k-1}, x_k]} f(x), \sup_{x \in [x_{k-1}, x_k]} f(x)]$. Since

$$\sum_{k=1}^{n} (F(x_k) - F(x_{k-1})) = F(b) - F(a)$$

the result follows by taking limits of finer and finer partitions in (2). □

Jargon

- An antiderivative of f is a function whose derivative is f.
- Sometimes this is called a primitive of f or the indefinite integral of f.
- Denoted $\int f$ (where the limits of integration are not specified). When the limits of integration are specified, $\int_a^b f$ is a number, called the definite integral of f on the interval $[a, b]$.

Improper Integrals

Extend definition to unbounded intervals by taking limits.

Properties of Integrals

1. For $\lambda, \mu \in \mathbb{R}$.

$$\int_a^b [\lambda f(x) + \mu g(x)] dx \;=\; \lambda \int_a^b f(x) dx \;+\; \mu \int_a^b g(x) dx$$

2. If f and g are continuous on $[a, b]$ with antiderivatives F and G, then

$$\int_a^b f(x) G(x) \;=\; [F(x)G(x)]_a^b \;-\; \int_a^b F(x) g(x) dx$$

3. If π is strictly monotonic on $[a, b]$ and f is continuous on an open interval containing $\pi([a, b])$. Then

$$\int_{\pi(a)}^{\pi(b)} f(x) dx \;=\; \int_a^b f(\pi(t)) \pi'(t) dt$$

4. If $f : [a, b] \longrightarrow \mathbb{R}$ is continuous, then there exists $t \in [a, b]$ such that
$$\int_a^b f(x)dx = f(t)(b - a).$$

1. A direct consequence of the definitions.
2. $$\frac{d}{dx}(FG) = fG + Fg$$
$$\Longrightarrow fG = \frac{d}{dx}(FG) - Fg$$
3. Think of $\pi'(t)(x_{i+1}) - x_i)$ as approximately $\pi(x_{i+1} - \pi(x_i)$.

4. This result is a simple consequence of the intermediate value theorem. By the definition of the integral it cannot be that

$$\int_a^b f(x)dx > f(t)(b-a)$$

for all $t \in [a, b]$ nor can it be that

$$\int_a^b f(x)dx < f(t)(b-a)$$

for all $t \in [a, b]$. Hence

$$\int_a^b f(x)dx - f(t)(b-a)$$

is a continuous function of t that changes sign on $[a, b]$ so it must equal zero for some value of t.

Names for the results

1. The second formula is called integration by parts and it comes up a lot.
2. The third formula is the change of variables formula.
3. The fourth result is called the mean value theorem for integrals and states that at some point in an interval a function must take on the average value of the function on the interval.

Computing Integrals

Since every continuous function is integrable, we know a lot of functions that are integrable. Coming up with formulas for the integrals is not easy in practice. Polynomials are easy. One can integrate the exponential function ($\int e^x = e^x$) and the identity

$$\int_1^x \frac{1}{t} dt = \log x$$

is often taken to be the definition of the log function. Other than that, the change of variables formula and integration by parts are the primary methods we have to find integrals.

Examples

Let $F' = f$.

$$\int_a^b xf(x)dx = bF(b) - aF(a) - \int_a^b F(x)dx$$

In applications, F is a CDF, f a density, $F(b) = 1$, $F(a) = 0$.
LHS is mean of distribution. So mean in

$$b - \int_a^b F(x)dx$$

.

$$\int_{-\infty}^{\infty} x^n e^{-x} dx = n \int_{-\infty}^{\infty} x^n e^{-x} dx$$

(I simplified using $\lim_{x \to \infty} x^n e^{-x} = 0$).