Econ 205 - Slides from Lecture 1

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Warning

I can't start without assuming that something is "common knowledge." You can find basic definitions of Sets and Set Operations (Union, Intersection, . . .) in the lecture notes.

Functions

Definition

A **function** f from a set X to a set Y is a specification (a mapping) that assigns to each element of X exactly one element of Y. Typically we express that f is such a mapping by writing $f : X \to Y$. The set X (the points one "plugs into" the function) is called the **domain** of the function, and the set Y (the items that one can get out of the function) is called the function's **codomain** or **range**.

We write $f(x)$ as the point in Y that the function associates with $x \in X$.

Definition

The **image** of a function f from a set X to a set Y is $\{y \in Y : f(x) = y \text{ for some } x \in X\}$. We denote this set by $f(X)$.

Definition

The **inverse image** of a set $W \subset Y$ as

$$f^{-1}(W) \equiv \{x \in X \mid f(x) \in W\}.$$

This is the set of points in X that map to points in W.

Note that the inverse image does not necessarily define a function from Y to X, because it could be that there are two distinct elements of X that map to the same point in Y.

More Definitions

Definition

A function $f : X \to Y$ is **onto** if $f(X) = Y$.

Definition

Consider a function $f : X \to Y$. This function is said to be **one-to-one** if $f(x) = f(x')$ implies that $x = x'$. An equivalent condition is that for every two points $x, x' \in X$ such that $x \neq x'$, it is the case that $f(x) \neq f(x')$.

Definition

A function $f : X \to Y$ is called **invertible** if the inverse image mapping is a function from Y to X. Then, for every $y \in Y$, $f^{-1}(y)$ is defined to be the point $x \in X$ for which $f(x) = y$.

Theorem

A function is invertible if and only if it is both one-to-one and onto.

Algebra of Functions

Definition

Suppose we have functions $g: X \to Y$ and $f: Y \to Z$. Then the **composition function** $f \circ g$ is a mapping from X to Z (that is, $f \circ g :: X \to Z$). Writing $h = f \circ g$, this function is defined by $h(x) \equiv f(g(x))$ for every $x \in X$.

When the range has special structure (as it will in almost all applications), there are other algebraic operations:

Definition

If $f, g\colon X \to Y$, then we can form new functions from $X \to Y$

- The **sum** (denoted $f + g$) and defined by $(f + g)(x) = f(x) + g(x)$.
- **Multiplication by a constant** (denoted λf) and defined by $(\lambda f)(x) = \lambda f(x)$.
- The **product** (denoted fg) and defined by $(fg)(x) = f(x)g(x)$.
- The **quotient** (denoted f/g) and defined by $(f/g)(x) = f(x)/g(x)$.

These definitions make sense exactly when it is possible to add, multiply by a constant, multiply, or divide elements of Y.
When is that possible?

The Real Line

The set of **real numbers**, denoted by $\mathbb{R}$. A class in "real analysis" would construct the real line from more basic sets – first integers, then ratios of integers (called rational numbers), then (in one construction) limits of rational numbers. We will skip this.
For $a, b \in \mathbb{R}$ where $a \leq b$

- Closed Interval: $[a, b] \equiv \{x : a \leq x \leq b\}$
- Half-open interval: $(a, b] \equiv \{x : a < x \leq b\}$ or $[a, b) \equiv \{x : a \leq x < b\}$
- Open interval: $(a, b) \equiv \{x :| a < x < b\}$

Can have $a = -\infty$ or $b = \infty$.
For example, $(-\infty, b) \equiv \{x : x < b\}$.

Bounds

Definition

Take $X \subset \mathbb{R}$. $a \in \mathbb{R}$ is an *upper bound* for X if

$$a \geq x, \quad \forall x \in X$$

$b \in \mathbb{R}$ is a *lower bound* for X if

$$b \leq x, \quad \forall x \in X$$

The set X is bounded" if "X is bounded from above" <u>and</u> "X is bounded from below."

Definition

$a \in \mathbb{R}$ is a *least upper bound of* X, or the *supremum* of X, if $a \geq x, \forall x \in X$ (a is an upper bound), and if a' is also an upper bound then $a' \geq a$. We write $a = \sup X$.

Definition

$b \in \mathbb{R}$ is a *greatest lower bound of* X, or the *infimum* of X, if $b \leq x, \forall x \in X$ (b is an lower bound), and if b' is also an lower bound then $b' \leq b$. We write

$$b = \inf X$$

sup and inf need not be in set $((0, 1))$.

Theorem

Any set $X \subset \mathbb{R}$, ($X \neq \phi$), that has an upper bound, has a least upper bound (sup).

Definition

$$\max X \equiv \text{a number } a \text{ such that}$$
$$a \in X \text{ and } a \geq x, \forall x \in X$$

and

$$\min X \equiv \text{a number } b \text{ such that } b \in X \text{ and } b \leq x, \forall x \in X$$

Max and min do not always exist even if the set is bounded, but the sup and the inf do always exist if the set is bounded. If $\sup X \in X$, then $\max X$ exists and is equal to $\sup X$.

Simple Result

Theorem
If max X exists then

$$max\ X = sup\ X$$

Let $a \equiv \max X$
We will demonstrate that a is $\sup X$.
To do this we must show that:
(i) a is an upper bound
(ii) Every other upper bound a' satisfies $a' \geq a$
so
(i) a is an upper bound on X since by the definition of max

$$a \geq x, \quad \forall x \in X$$

(ii) Consider any other upper bound a' of the set X such that $a' \neq a$.
Since a' is an upper bound, and $a \in X$ (by the fact that $a = \max X$), we must have $a' > a$.

Sequences

Let $\mathbb{P}$ denote the positive integers.

Definition

A *sequence* is a function f from $\mathbb{P}$ to $\mathbb{R}$.

$$f : \mathbb{P} \longrightarrow \mathbb{R}$$

If $f(n) = a_n$, for $n \in \mathbb{P}$, we denote the sequence f by the symbol $\{a_n\}_{n=1}^{\infty}$, or sometimes by $\{a_1, a_2, a_3, \ldots\}$. The values of f, that is, the elements a_n, are called the *terms* of the sequence.

Examples

1. $a_n = n$, $\{a_n\}_{n=1}^{\infty} = \{1, 2, 3, \ldots\}$
2. $a_1 = 1$, $a_{n+1} = a_n + 2, \forall\, n > 1$, $\{a_n\}_{n=1}^{\infty} = \{1, 3, 5, \ldots\}$
3. $a_n = (-1)^n, n \geq 1,\{a_n\}_{n=1}^{\infty} = \{-1, 1, -1, 1, \ldots\}$

Limits of Sequences

Definition

[Convergent Sequence]
A sequence $\{a_n\}$ is said to *converge*, if there is a point $a \in \mathbb{R}$ such that for every $\varepsilon > 0 \; \exists \, N \in \mathbb{P}$ such that if $n \geq N$, then $\mid a_n - a \mid < \varepsilon$.

In this case we also say that $\{a_n\}$ converges to a and we write

$$a_n \longrightarrow a$$

or

$$\lim_{n \to \infty} a_n = a$$

If $\{a_n\}$ does not converge, it is said to *diverge*.

Properties

Theorem

Let $\{a_n\}$ be a sequence in $\mathbb{R}$.
If $b \in \mathbb{R}$, $b' \in \mathbb{R}$, and if $\{a_n\}$ converges to b and to b', then

$$b = b'$$

Proof.

Assume, in order to obtain a contradiction, that $b > b'$
Let $\varepsilon = \frac{1}{2}(b - b') > 0$.
By definition, there is N and N' such that
$\forall\, n \geq N$

$$\mid b - a_n \mid < \varepsilon$$

and $\forall\, n \geq N'$

$$\mid b' - a_n \mid < \varepsilon$$

This is impossible since it implies that

$$\varepsilon = b - b' \leq \mid b - a_n \mid + \mid b' - a_n \mid < \varepsilon$$

Subsequence

Definition

A subsequence of the sequence described by f from $\mathbb{P}$ to $\mathbb{R}$. is the sequence given by a function

$$f \circ g : \mathbb{P} \longrightarrow \mathbb{R}$$

where

$$g : \mathbb{P} \longrightarrow \mathbb{P}$$

and $g(m) > g(n)$ whenever $m > n$.

More commonly if $\{a_n\}$ is a sequence and $\{n_k\}$ of positive integers, such that $n_1 < n_2 < n_3 < \cdots$, then the sequence $\{a_{n_i}\}$ is called a *subsequence* of $\{a_n\}$. If $\{a_{n_i}\}$ converges, its limit is called a *subsequential limit* of $\{a_n\}$.

Monotonicity

Definition

A sequence of real numbers $\{a_n\}$ of real numbers is said to be

1. *monotonically increasing* if

$$a_n \leq a_{n+1} \qquad (n = 1, 2, 3, \dots)$$

2. *monotonically decreasing* if

$$a_n \geq a_{n+1} \qquad (n = 1, 2, 3, \dots)$$

We say strictly increasing/decreasing if the above weak inequalities are replaced with strict inequalities.

Definition

A function f is said to be

1. *monotonically increasing* on (a, b) if

$$a < x < y < b \quad \Longrightarrow \quad f(x) \le f(y)$$

2. *monotonically decreasing* on (a, b) if

$$a < x < y < b \quad \Longrightarrow \quad f(x) \ge f(y)$$

A function f is strictly increasing/decreasing if the above weak inequalities are replaced with strict inequalities.

Facts

1. $\{a_n\}$ converges to b if and only if every subsequence of $\{a_n\}$ converges to b.
2. If $\{a_n\}$ and $\{b_n\}$ are sequences, and

$$\lim_{n\to\infty} a_n = a$$

$$\lim_{n\to\infty} b_n = b,$$

then

2.1 $\lim_{n\to\infty}(a_n + b_n) = a + b$
2.2 $\lim_{n\to\infty} ca_n = ca, c \in \mathbb{R}$
2.3 $\lim_{n\to\infty} a_n b_n = ab$
2.4 $\lim_{n\to\infty}(a_n)^k = a^k$
2.5 $\lim_{n\to\infty} \frac{1}{a_n} = \frac{1}{a}$ provided $a_n \neq 0 (n = 1, 2, 3, \dots)$, and $a \neq 0$
2.6 $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{a}{b}$ provided $b_n \neq 0 (n = 1, 2, 3, \dots)$, and $b \neq 0$

Definition

We say the sequence $\{a_n\}$ is bounded above if $\exists\ \overline{m} \in \mathbb{R}$ such that

$$a_n \leq \overline{m}, \quad \forall\, n \in \mathbb{P}$$

We say the sequence $\{a_n\}$ is bounded below if $\exists\ \underline{m} \in \mathbb{R}$ such that

$$a_n \geq \underline{m}, \quad \forall\, n \in \mathbb{P}$$

Theorem

If $\{a_n\}$ converges then $\{a_n\}$ is bounded.

Proof.

Suppose $a_n \longrightarrow a$. There is an integer N such that $n > N$ implies $\mid a_n - a \mid < 1$. Put

$$r = max\{1, \mid a_1 - a \mid, \ldots, \mid a_N - a \mid\}$$

Then $\mid a_n - a \mid \leq r$ for $n = 1, 2, 3, \ldots$. □

Theorem

Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be sequences such that

$$a_n \leq b_n \leq c_n, \qquad \forall\, n \in \mathbb{R}$$

and we have that both

$$a_n \longrightarrow a, \qquad c_n \longrightarrow a$$

$$\Longrightarrow b_n \longrightarrow a$$

Theorem

A monotone sequence is convergent if and only if it is bounded.

Proof.

1. We already showed that <u>any</u> convergent sequence is bounded in Theorem 23. So a monotone sequence is bounded.
2. Suppose $\{a_n\}$ is bounded. Thus by the set $X \equiv \{a_n \mid n \in \mathbb{P}\}$ has a sup which we will denote $\bar{a}$. We must show that $\bar{a}$ is the limit of $\{a_n\}$. We need to show that $\forall \varepsilon > 0, \exists N \in \mathbb{P}$ such that

   $$\mid \bar{a} - a_n \mid < \varepsilon, \quad \forall n \geq N$$

   Take any $\varepsilon > 0$. Since $\bar{a} - \varepsilon$ is not an upper bound of X, $\exists N$ such that $a_N > \bar{a} - \varepsilon$. Since $\{a_n\}$ is increasing, for every $n \geq N$ we have

   $$\bar{a} - \varepsilon < a_N \leq a_n \leq \bar{a}$$

   since we also have that $a_n \leq \bar{a}$, $\forall n$.
   Thus,

   $$\mid a_n - \bar{a} \mid < \varepsilon, \qquad \forall n \geq N$$

□

Theorem

Every real sequence has a monotone subsequence.

Proof.

Take a sequence $\{x_n\}$. We must find a subsequence $\{x_{n_k}\}$ that is either increasing or decreasing.
There are two cases:

1. $\{x_n \mid n \geq N\}$ has a maximum for all N.
   And if this is true we should be able to form a decreasing sequence
2. $\{x_n \mid n \geq N\}$ doesn't have a maximum for some $N \in \mathbb{P}$

Fill in details! □

Theorem (Bolzano-Weierstrass)

Every bounded, real sequence has a convergent subsequence.

Proof.

We know that every real sequence has a monotone subsequence and that every bounded sequence is bounded. □

Analysis Definitions

1. $X \subset \mathbb{R}$ is *compact* if every sequence in X has a convergent subsequence in X.
2. X is called a *closed* set if every sequence that converges in X converges to a point in X.
3. X is called *open* if for every $x \in X$, there exists $\varepsilon > 0$ such that

$$(x - \varepsilon, x + \varepsilon) \subset X$$

Limits of Functions

Take any function f,

$$f : X \longrightarrow \mathbb{R}$$

for $X = (c, d)$ (so note that it is an open set) and take $a \in X$.

Definition

y is called the *limit of f from the right at a* (right hand limit) if, for every $\varepsilon > 0$, there is a δ such that

$$0 < x - a < \delta \quad \Longrightarrow \mid f(x) - y \mid < \varepsilon$$

this is also denoted as

$$y = \lim_{x \to a^+} f(x)$$

Note: $x > a$, so x is to the right of a.

Limits of Functions

Take any function f,

$$f : X \longrightarrow \mathbb{R}$$

for $X = (c, d)$ (so note that it is an open set) and take $a \in X$.

Definition

y is called the *limit of f from the left at a* (left-hand limit) if, for every $\varepsilon > 0$, there is a δ such that

$$0 < a - x < \delta \quad \Longrightarrow | f(x) - y | < \varepsilon$$

this is also denoted as

$$y = \lim_{x \to a^-} f(x)$$

Note: $x < a$, so x is to the left of a.

LIMIT

Definition

y is called the *limit of f at a* if

$$y = \lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$$

and we write

$$y = \lim_{x \to a} f(x)$$

y is the limit of f at a if for every $\varepsilon > 0$, there is a $\delta > 0$ such that

$$0 < |\ x - a\ | < \delta \qquad \Longrightarrow \qquad |\ f(x) - y\ | < \varepsilon$$

Note: The definition of the limit of a function at the point a does not require the function to be defined at a.

Examples

Consider the function

$$f(x) = \begin{cases} 0, & \text{if } x \neq 0, \\ 1, & \text{if } x = 0. \end{cases}$$

$$\begin{aligned} \lim_{x \to 0^-} f(x) &= 0 = \lim_{x \to 0^+} f(x) \\ \Rightarrow \lim_{x \to 0} f(x) &= 0 \end{aligned}$$

But the value of the function at the point 0 is not equal to the limit. The example shows that it is possible for $\lim_{x \to a} f(x)$ to exist but for it to be different from $f(a)$. Since you can define the limit without knowing the value of $f(a)$, this observation is mathematically trivial. It highlights a case that we wish to avoid, because we want a function's value to be approximated by nearby values of the function.

Properties

Theorem

Limits are unique. That is, if $\lim_{x \to a} f(x) = L$ *and* $\lim_{x \to a} f(x) = L'$*, then* $L = L'$*.*

Proof.

Assume that $L \neq L'$ and argue to a contradiction. Let $\varepsilon = \mid L - L' \mid /2$. Given this ε let $\delta^* > 0$ have the property that $\mid f(x) - L \mid$ and $\mid f(x) - L' \mid$ are less than ε when $0 < \mid x - a \mid < \delta^*$. (This is possible by the definition of limits.) Since

$$\mid f(x) - L \mid + \mid f(x) - L' \mid \geq \mid L - L' \mid \tag{1}$$

it follows that $\varepsilon \geq \mid L - L' \mid$, which is not possible. □

The theorem simply states that the function cannot be close to two different things at the same time. Inequality (inequality (1)) is called the triangle inequality. It says (in one dimension) that the distance between two numbers is no larger that the distance between the first number and a third number plus the distance between the second number and the third number. In general, it says that the shortest distance between two points in a straight line.

Theorem

If a limit exists, then both limit from the left and limit from the right exist (and they are equal) and, conversely, if both limit from right and limit from left exist, then the limit exists.

Theorem

If f and g are functions defined on a set S, $a \in (\alpha, \beta) \subset S$ and $lim_{x \to a} f(x) = M$ and $lim_{x \to a} g(x) = N$, then

1. $lim_{x \to a} (f + g)(x) = M + N$
2. $lim_{x \to a} (fg)(x) = MN$
3. $lim_{x \to a} \left(\frac{f}{g}\right)(x) = M/N$ *provided* $N \neq 0$.

For the first part, given $\varepsilon > 0$, let $\delta_1 > 0$ be such that if $0 <| x - a |< \delta_1$ then $| f(x) - M |< \varepsilon/2$ and $\delta_2 > 0$ be such that if $0 <| x - a |< \delta_2$ then $| g(x) - N |< \varepsilon/2$. This is possible by the definition of limit. If $\delta = \min\{\delta_1, \delta_2\}$, then $0 <| x - a |< \delta$ implies

$$| f(x) + g(x) - M - N | \leq | f(x) - M | + | g(x) - N | < \varepsilon/2 + \varepsilon/2 < \varepsilon.$$

The first inequality follows from the triangle inequality while the second uses the definition of δ. This proves the first part of the theorem.

For the second part, set δ_1 so that if $0 <\mid x - a \mid< \delta_1$ then $\mid f(x) - M \mid< \sqrt{\varepsilon}$ and so on.
For the third part, note that when $g(x)$ and N are not equal to zero

$$\frac{f(x)}{g(x)} - \frac{M}{N} = \frac{g(x)\,(f(x) - M) + f(x)\,(N - g(x))}{g(x)N}.$$

So given $\varepsilon > 0$, find δ so small that if $0 <\mid x - a \mid \delta$, then $\frac{|f(x)|}{|g(x)N|} < \frac{2|M|}{N^2}$, $\mid f(x) - M \mid< N\varepsilon/2$ and $\mid g(x) - N \mid< \frac{N^2}{4|M|}\varepsilon/2$.
This is possible provided that $N \neq 0$.

Definition

We say $f : X \to \mathbb{R}$ is *continuous at a* if

$$f(a) = \lim_{x \to a} f(x)$$

- f is said to be continuous at the point a. If f is continuous at every point of X, then f is said to be *continuous* on X.
- the limit exists and the limit of the function is equal to the function of the limit. That is,

  $$\lim_{x \to a} f(x) = f(\lim_{x \to a} x).$$

  I state and prove a formal version of this below.
- The function has to be defined at a for the limit to exist.
- In order for a function to be continuous at a, it must be defined "in a neighborhood" of a – that is, the function must be defined on an interval (α, β) with $a \in (\alpha, \beta)$. We extend the definition to take into account "boundary points" in a natural way: We say that f defined on $[a, b]$ is continuous at a (resp. b) if $\lim_{x \to a^+} f(x) = f(a)$ (resp. $\lim_{x \to b^-} f(x) = f(b)$).

Lemma

$f : X \longrightarrow \mathbb{R}$ is continuous at a if and only if for every $\varepsilon > 0$, there is a $\delta > 0$ such that

$$0 <\mid a - x \mid< \delta \qquad \Longrightarrow \qquad \mid f(x) - f(a) \mid< \varepsilon$$

Note we could also write the conclusion as:

$$\mid a - x \mid< \delta \qquad \Longrightarrow \qquad \mid f(x) - f(a) \mid< \varepsilon$$

Sequential Continuity

Theorem

$f : X \longrightarrow \mathbb{R}$ is continuous at a if and only if for every sequence $\{x_n\}$ that converges to a, $\lim_{n\to\infty} f(x_n) = f(a)$.

Proof.

Use the formulation of continuity in the previous lemma. Let $\varepsilon > 0$ be given. Let $\delta > 0$ correspond to that ε. Take a sequence $\{x_n\}$ that converges to a. We know that there is N such that if $n > N$, then $| a - x_n | \delta$ implies that $| f(x) - f(a) |< \varepsilon$.

Conversely, suppose that f is continuous at a and $\{x_n\}$ converges to a. We want to show that $\{f(x_n)\}$ converges to $f(a)$. Let $\varepsilon > 0$ be given. We know that there exists $\delta > 0$, such that if $| a - x | \delta$, then $| f(a) - f(x) | \varepsilon$. From the definition of the convergence of $\{x_n\}$ there is an N such that $n > N$ implies $| a - x_n |< \delta$. Hence $\{f(x_n)\}$ converges to $f(a)$.

□

Algebra of Continuity

The sums, products, and ratios of continuous functions are continuous (in the last case the denominator of the ratio must be non-zero in the limit).

Theorem

For

$$g : X \longrightarrow Y$$

and

$$f : Y \longrightarrow \mathbb{R}$$

where both X and Y are open intervals of $\mathbb{R}$, if g is continuous at $a \in X$ and if f is continuous at $g(a) \in Y$, then

$$f \circ g : X \longrightarrow \mathbb{R}$$

is continuous at a.

Let $\varepsilon > 0$ be given. Since f is continuous at $g(a)$, there exists $\gamma > 0$ such that

$$| f(y) - f(g(a)) | < \varepsilon$$

if

$$| y - g(a) | < \gamma$$

and $y \in g(X)$.
Since g is continuous at a, there exist $\delta > 0$ such that

$$| g(x) - g(a) |< \gamma$$

if

$$| x - a |< \delta$$

and $x \in X$.
It follows that

$$| (g(x)) - f(g(a)) |< \varepsilon$$

if

$$| x - a |< \delta$$

and $x \in X$. Thus $f \circ g$ is continuous at a.

Use of Algebra of Continuity

These functions are continuous:

1. Constant functions ($f(x) \equiv c$).
2. Linear functions ($f(x) \equiv ax$).
3. Affine functions ($f(x) \equiv ax + b$).
4. Polynomials of degree n ($f(x) \equiv a_0 + a_1 x + a_2 x^2 + \ldots a_n x^n$, $a_n \neq 0$)
5. Rational functions ($f(x) \equiv P(x)/Q(x)$ for polynomials P and Q) at any point a where $Q(x) \neq 0$.

Continuous Image of Closed Bounded Interval

Theorem

The continuous image of a closed and bounded interval is a closed and bounded interval. That is, if

$$f : [a, b] \longrightarrow \mathbb{R}$$

is continuous, then there exists $d \geq c$ such that $f([a, b]) = [c, d]$.

The assumptions in the theorem are important. If f is not continuous, then there generally nothing that can be said about the image. If the domain is an open interval, the image could be a closed interval (it is a point of f is constant) or it could be unbounded even if the interval is finite (for example if $f(x) = 1/x$ on $(0, 1)$).

Existence of Max

Definition

We say that $x^* \in X$ maximizes the function f on X if

$$f(x^*) \geq f(x),$$

for every $x \in X$.

Similar definition for min of a fuction.

When the domain of f is a closed, bounded interval, the image of f is a closed, bounded interval, f attains both its maximum (the maximum value is d) and its minimum. This means that if a function is continuous and it is defined on a "nice" domain, then it has a maximum.

Intermediate Value Theorem

Theorem

If $f : [a, b] \longrightarrow \mathbb{R}$ is continuous, and z is between $f(a)$ and $f(b)$, then there exists $z \in [a, b]$ such that $f(z) = [a, b]$

In the statement of the theorem, "between" means either $z \in [f(a), f(b)]$ (when $f(a) \leq f(b)$ or $z \in [f(b), f(a)]$ if $f(b) < f(a)$.

Proof.

$f([a, b])$ is an interval. Hence if two points are in the image, then all points between these two points are in the image. □

Theorem **??** is a method of solving showing that equations have solutions. It is common to conclude that there must be a c for which the continuous function f is zero ($f(c) = 0$) from the observation that sometimes f is positive and sometimes f is negative. The most important existence theorems in micro (the existence of a market clearing price system, for example) follow from (harder to prove) versions of this result.

A Baby Fixed-Point Theorem

The point $x^* \in S$ is called a *fixed point* of the function $f : S \longrightarrow S$ if $f(x^*) = x^*$.

Theorem

Let $S = [a, b]$ be a closed bounded interval and $f : S \longrightarrow S$ a continuous function. There exists $x^* \in S$ such that $f(x^*) = x^*$.

Proof.

Consider the function $h(x) = f(x) - x$. Since $f(a) \geq a$, $h(a) \geq 0$. Since $f(b) \leq b$, $h(b) \leq 0$. Since h is a continuous function on a closed bounded interval, there must be an x^* such that $h(x^*) = 0$. It is clear that for this value, $f(x^*) = x^*$. □

Assumptions are important

1. There are functions that don't attain max (due to "bad" domain or failure of continuity).
2. There are functions that don't have fixed points.

Monotonicity

Definition

A function f is said to be

1. *monotonically increasing* on (a, b) if

$$a < x < y < b \quad \Longrightarrow \quad f(x) \leq f(y)$$

2. *monotonically decreasing* on (a, b) if

$$a < x < y < b \quad \Longrightarrow \quad f(x) \geq f(y)$$

Note we say a function f is <u>strictly</u> increasing/decreasing if the above weak inequalities are replaced with strict inequalities.