

# Economics 205 Final Examination Fall 2010

## Instructions.

1. You have three hours to complete this examination.
2. You may use scratch paper, but please write your final answers (including your complete arguments) on these sheets.
3. You may use one page of notes.
4. Try to answer all eight problems. (Read all of the questions now and start on the ones that seem easiest.)
5. Do not be intimidated by the multi-part questions. In most cases, it is possible to answer the later parts without answering the earlier parts.
6. Think before you write. Doing so may save time and needless computations.
7. Read each question carefully and answer the question that I ask. If you are uncertain about how to interpret the question, please ask me for clarification.
8. Make your answers as complete and rigorous as possible. In particular, give reasons for your computations, prove your assertions. You may use any result presented in class or supplementary material to support your arguments, provided that state clearly the result that you are using and confirm that all necessary assumptions hold.
9. Informal and intuitive arguments are better than no arguments.
10. The table below gives the point values for each question, allocate your time appropriately.
11. I plan to return the graded exams to your mailboxes on Monday afternoon (there is no chance that they will arrive earlier).

	Score	Possible
I		150
II		90
III		150
IV		170
V		100
VI		180
VII		200
VIII		160
Exam Total		1200
Course Total		1200
Grade in Course		

1. In each part, determine at which points the derivative of the function  $h$  exists. When it does exist, compute it. When it does not exist, explain why it does not exist.

(a)  $h(x) = \log(1 + \log(1 + x))$ .

(b)  $h(x) = (e^{\log x})^2$ .

(c)  $h(x) = \int_0^x f(y)dy$  for a continuous function  $f$ .

(d)  $h(x) = \int_0^x f(x)dy$  for a continuous function  $f$ .

(e)  $h(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ x^2 \log x & \text{if } x > 0 \end{cases}$

2. Use integration by parts to compute  $\int_1^2 x^2 \log x \, dx$ .

3. Consider the function  $f(x, y) = \left(\frac{1}{x^2} - \frac{1}{y}\right)$  defined on  $\{(x, y) : x, y > 0\}$ .
- (a) Graph  $\{(x, y) : x, y > 0, f(x, y) = -1\}$ .
  - (b) Find an equation of the hyperplane tangent to the graph of  $f(x, y) = z$  at the point  $(x, y, z) = (1, 1, 0)$ .
  - (c) Decide whether or not  $f(\cdot)$  is homogeneous. If  $f(\cdot)$  is homogeneous, then determine its degree of homogeneity and explicitly verify Euler's Theorem.
  - (d) Find the directional derivative of  $f$  in the direction  $\mathbf{w} = \frac{1}{\sqrt{2}}(1, 1)$  at the point  $(x, y) = (.5, 1)$ .
  - (e) Let  $g(u, v) = (u^2 + v^2 + 1, u + 1)$ . Use the chain rule to compute all partial derivatives of  $f \circ g$  when  $u = v = 0$ .

4. Consider the following matrices:

$$\text{(a) } \mathbf{A} = \begin{pmatrix} 5 & 3 \\ 4 & 3 \end{pmatrix}$$

$$\text{(b) } \mathbf{A} = \begin{pmatrix} 6 & 9 \\ 2 & 3 \end{pmatrix}$$

$$\text{(c) } \mathbf{A} = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & \frac{2}{3} \end{pmatrix}.$$

- 3.1 Say whether  $\mathbf{A}^{-1}$  exists and, if so, find  $\mathbf{A}^{-1}$ .
- 3.2 State whether the quadratic form  $\mathbf{x}^t \mathbf{A} \mathbf{x}$  is positive definite, positive semi-definitive, indefinite, negative semi-definite, or negative definite.
- 3.3 State whether the matrix is diagonalizable.
- 3.4 For matrix  $\mathbf{b}$ , find the eigenvalues and exhibit an invertible square matrix  $\mathbf{P}$  and a diagonal matrix  $\mathbf{D}$  such that  $\mathbf{A} = \mathbf{P} \mathbf{D} \mathbf{P}^{-1}$ .

5. A piece of cheese is located at  $(12, 10)$  in  $\mathbb{R}^2$ . A mouse is at  $(4, -2)$  and is running up the line  $y = -5x + 18$ . At the point  $(a, b)$  the mouse starts getting farther from the cheese rather than closer to it. What is  $a + b$ ?

6. An agent has utility function  $u(x)$  where  $x$  is income. The agent has initial wealth  $w$ . With probability  $p$  the agent suffers a loss of  $l$  dollars, reducing her wealth to  $w - l$ . The agent's *expected utility* is  $pu(w - l) + (1 - p)u(w)$ . The agent's *certainty equivalent*  $C$  is implicitly defined by the equation

$$u(C) = pu(w - l) + (1 - p)u(w). \quad (1)$$

- (a) Show that if  $u$  is continuous, there exists a certainty equivalent. That is, there is a solution to equation (1).
- (b) Show that if  $u$  is strictly increasing, then there exists a unique certainty equivalent. That is, there is one and only one  $C$  that solves equation (1).
- (c) Suppose that for given values  $(p_0, w_0, l_0)$  equation (1) has a solution  $C_0$ . State conditions on  $u$  and its derivatives under which you can locally solve equation (1) for  $C$  as a differentiable function  $C = g(p, w, l)$  in a neighborhood of  $(p_0, w_0, l_0)$  with  $g(p_0, w_0, l_0) = C_0$ . Write down a formula for  $Dg(p_0, w_0, l_0)$ .
- (d) State economically plausible conditions under which  $g$  is decreasing in  $p$ .

7. A firm that uses two inputs to produce output has the production function  $3x^{1/3}y^{1/3}$ , where  $x$  is the amount of input 1 and  $y$  is the amount of input 2. The price of output is 1. The cost of the inputs are  $wx$  and  $wy$ . The firm is constrained by the government to use no more than 1000 units of input 1.
- (a) What is the profit maximizing input combination for the firm?
  - (b) What is the most that the firm is willing to pay to have the right to increase the limit on input 1 by a tiny amount (from 1000 to  $1000 + \Delta$  units),  $\Delta > 0$ ?

8. Decide whether each of the statements below is true. If the statement is true, then prove it. If the statement is false, then given a counterexample.

- (a) For any matrix  $\mathbf{A}$ ,  $\mathbf{A}^t \mathbf{A}$  is a positive semi-definite matrix.
- (b) If  $\mathbf{A}$  is a symmetric matrix that satisfies  $\mathbf{A}^3 = \mathbf{I}$ , then  $\mathbf{A} = \mathbf{I}$ .
- (c) If  $f : \mathbb{R} \rightarrow \mathbb{R}$  is a continuous function that satisfies  $f(x) = -f(-x)$ , then  $f(0) = 0$ .
- (d) If  $f : \mathbb{R} \rightarrow \mathbb{R}$  is a twice continuously differentiable function that satisfies  $f(0) = f(1) = f(2)$ , then there exists  $c \in [0, 2]$  such that  $f''(c) = 0$ .