

Economics 205, Fall 2008: Notes on Returns to Scale

A function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  exhibits **increasing returns to scale** if for all  $a > 1$  and  $x \in \mathbb{R}^n$ ,  $af(x) \leq f(ax)$ .

A function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  exhibits **constant returns to scale** if for all  $a > 1$  and  $x \in \mathbb{R}^n$ ,  $af(x) = f(ax)$ .

A function  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  exhibits **decreasing returns to scale** if for all  $a > 1$  and  $x \in \mathbb{R}^n$ ,  $af(x) \geq f(ax)$ .

What is the importance of  $a > 1$  in these definition? It correctly treats the condition of one describing raising the scale of production. If doubling inputs ( $a = 2$ ) leads to less than a doubling of output when the input vector is  $x$ , then it is obvious (to anyone but me, when I was in class today) that cutting the inputs in half when the input vector is  $2x$  will leave you with more than half of the output you receive when from  $2x$ . That is, unless you have constant returns to scale, it does not make sense to impose  $f(ax) > af(x)$  for all  $x$  and all  $a$  ( $a$  both less than and greater than 1).

What is the relationship between returns to scale and concavity? Suppose  $f(0) = 0$ . If  $f$  is concave, then

$$f(\alpha y) = f(\alpha y + (1 - \alpha)0) \geq \alpha f(y) + (1 - \alpha)f(0) = \alpha f(y) \quad (1)$$

for all  $\alpha \in (0, 1)$ . Consequently,

$$f(\alpha y) \geq \alpha f(y)$$

and, letting  $a = 1/\alpha$  and  $x = \alpha y$

$$af(x) \geq f(ax)$$

for all  $a > 1$  and  $x \in \mathbb{R}^n$ . Consequently, any concave function satisfying  $f(0) = 0$  exhibits decreasing returns to scale.<sup>1</sup> Similarly, any convex function satisfying  $f(0) = 0$  exhibits increasing returns to scale and any linear function (which automatically satisfies  $f(0) = 0$ ) exhibits constant returns to scale.

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<sup>1</sup>Note that this argument is still true if  $f(0) \geq 0$ ; when  $f(0) > 0$  the last equation in (1) should be replaced by a greater than or equal sign.