

Economics 205: Supplementary Problems

1. Prove by mathematical induction:
 - (a) $1 + 2 + \cdots + n = n(n + 1)/2$.
 - (b) $1 + 3 + \cdots + (2n - 1) = n^2$.
 - (c) $1 + 4 + \cdots + n^2 = n(n + 1)(2n + 1)/6$.
 - (d) $1 + r + \cdots + r^{n-1} = (1 - r^n)/(1 - r)$.
2. Prove that if a and b are real numbers, then $|| a | - | b || \leq | a - b |$.
3. For each of the sequences below find (if it exists):
 - (a) the range of the sequence.
 - (b) an upper bound.
 - (c) a maximum.
 - (d) a supremum.
 - (e) a lower bound.
 - (f) a minimum.
 - (g) an infimum.
 - (h) a convergent subsequence.

The sequences are:

- (a) $a_n = 1$ for all n .
 - (b) $a_n = (-1)^n$ for all n .
 - (c) $a_n = 2^n$ for all n .
 - (d) $a_n = (-1/2)^n$ for all n .
 - (e) $a_n = 1/n$ for all n .
4. If $a_n = 1 + 1/2 + 1/3 + \cdots + 1/n$ for each n , prove that if k is fixed, then $\lim_{n \rightarrow \infty} | a_{n+k} - a_n | = 0$ but that the sequence does not satisfy the Cauchy criterion.
 5. Decide whether the following limits exist. If possible, compute the limit.
 - (a) $\lim_{n \rightarrow \infty} [(n^3 - 2)/6n^2]$.
 - (b) $\lim_{n \rightarrow \infty} [(n + 1)/(n - 1)]$.
 - (c) $\lim_{n \rightarrow \infty} x_n$, where $x_n = .6x_{n-1} + 5$, $x_1 = 0$.
 - (d) $\lim_{n \rightarrow \infty} [(7n^3 - 2)/(6n^4)]$.
 - (e) $\lim_{n \rightarrow \infty} [(n + 10)^{\cdot 5} - (n - 1)^{\cdot 5}]$.
 - (f) $\lim_{n \rightarrow \infty} x_n$, where $x_n = -.4x_{n-1} + 15$, $x_1 = 10$.
 6. $\lim_{n \rightarrow \infty} x_n = L$, then $\lim_{n \rightarrow \infty} | x_n | = | L |$.

7. Find the indicated limits.

- (a) $\lim_{x \rightarrow \infty} (\log x)/x$.
- (b) $\lim_{x \rightarrow \infty} [1 + (1/x)]^x$.
- (c) $\lim_{x \rightarrow \infty} [5x + 3]/[4x - 5]$.
- (d) $\lim_{x \rightarrow \infty} \exp(x^{-2})/x$.

8. Prove or give a counterexample: If the sequence $\{x_n\}_n^\infty$ has a bounded subsequence, then the sequence $\{x_n\}_n^\infty$ has a convergent subsequence.

9. Prove that if $\lim_{n \rightarrow \infty} x_n = L$, then $\lim_{n \rightarrow \infty} \lambda x_n = \lambda L$.

10. State which of the following functions are continuous at $x = 0$. Prove your answers.

(a)

$$f(x) = \begin{cases} \cos x & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

(b)

$$f(x) = \begin{cases} \sin x & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

(c)

$$f(x) = \begin{cases} \frac{\cos x}{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

(d)

$$f(x) = \begin{cases} \frac{\sin x}{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

(e)

$$f(x) = \begin{cases} x \sin(\frac{1}{x}) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

11. Which of the functions in question 10 are differentiable at $x = 0$? Prove your answers.

12. A twice continuously differentiable function $f(\cdot)$ from the real line to itself satisfies the following conditions: $f(1) = -1$, $f(2) = 3$, $f(3) = f(4) = 2$. Find the minimum number of times that $f(x) = 0$, $f'(x) = 0$, and $f''(x) = 0$.

13. Let $f(x) = 3x^3 + 1$.

(a) Is $f(\cdot)$ invertible? Explain. If f is invertible, then compute its inverse.

- (b) Graph $f(\cdot)$.
- (c) Find the equation of the line tangent to the graph of f at $x = 1$.
- (d) Find a second-order Taylor expansion of f around $x = 0$.
14. Repeat 13 for the functions below.
- (a) $f(x) = x \log(x + 1)$.
- (b) $f(x) = \exp(1 + x^2)$.
- (c) $f(x) = 3x - 1$.
15. Suppose that $f(\cdot)$ is function defined on \mathbb{R} that is differentiable at $x = 0$. If $f(0) = 0$ and $f'(0) > 0$, then prove that there exists an $\epsilon > 0$ such that if $\epsilon > y > 0$, then $f(y) > 0$.
16. Use Taylor's theorem to approximate $\log(\sqrt{1.1})$ to the nearest .001.
17. Let $f(\cdot)$ be a differentiable functions of one real variable. Calculate the derivative of the function $h(\cdot)$ defined in each of the problems below.
- (a) $h(x) = f(x)^a, a > 0$
- (b) $h(x) = e^{f(x)}$.
- (c) $h(x) = \log(f(x)^2 + 5)$.
18. Suppose that $f(\cdot)$ is a continuous real-valued function defined on the open interval $(-1, 1)$. Further suppose that $f(x)$ exists for all $x \neq 0$ and that $\lim_{x \rightarrow 0} f'(x)$ exists. Prove that $f(\cdot)$ is differentiable at $x = 0$ and that $f'(\cdot)$ is continuous at $x = 0$.
19. Give an example of a function that is continuously differentiable on the the interval $(-1, 1)$ but fails to be twice differentiable at zero. Prove your claims.
20. Let $f(x) = x^3 + ax^2 + bx + c$. Prove that there exists at least one value y such that $f(y) = 0$.
21. Use the definition of the derivative to prove that if $f(\cdot)$ and $g(\cdot)$ are differentiable at x , then $f \circ g(\cdot)$ is differentiable at x and $(f \circ g)'(x) = f(x)g'(x) + f'(x)g(x)$. [You can look this one up everywhere. Do it yourself instead.]
22. Show that the function $[(x^2 + 1)/(x + 2)] + [(x^4 + 1)/(x - 3)]$ is equal to zero for at least one value of x between -2 and 3.
23. Let

$$f(x) = \begin{cases} x & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}.$$

Find the set of points for which $f(\cdot)$ is continuous. Show that the function is not differentiable at any x .

24. Let $x = (1, 2, 3)$ and $y = (2, 3, 4)$.
- Write the equation of a line through x and y .
 - Find the equation of a plane that goes through x and y .
 - Find the equation of a line that passes through x and is orthogonal to the plane that you found in part b.
 - Find the distance from the origin to x , y , the lines that you found in parts a and c, and the plane that you found in part b.
25. Let $f(x) = x^2$ and $g(y_1, y_2) = y_1 + \log[(y_2)^2 + 3] + y_1 y_2$.
- Find the partial derivatives of $g(\cdot)$.
 - Find the partial derivatives of $f \circ g$ directly and by using the chain rule.
 - Which of the functions f , g , and $f \circ g$ are homogeneous?
 - Find the equation of a plane tangent to the graph of $g(\cdot)$ at the point $(y_1, y_2) = (1, 1)$.
 - If $F(x, y, z) = f[u(x, y, z), v(x, y, z)]$ evaluate $\frac{\partial F}{\partial x}$ at the point (x_0, y_0, z_0) if $u(x, y, z) = (x^2 - 4y)e^x$, $v(x, y, z) = xyz$, $f(u, v) = \log(u + v + 2)$, and $(x_0, y_0, z_0) = (2, 1, 0)$.
26. Prove that if $g(\cdot)$ is homogeneous of degree one, then $g(x, y)/(x + y)$ is homogeneous of degree zero. [Add any regularity conditions needed to prove this result.]
27. Prove that if $g(\cdot)$ is differentiable and homogeneous of degree n , then all of the partial derivatives of $g(\cdot)$ are homogeneous of degree $n - 1$. [Add any regularity conditions needed to prove this result.]
28. Give an example to show that there exists a function $f(\cdot)$ that is homogeneous of degree $n - 1$ and a function $F(\cdot)$ such that $F \equiv f$ but that $F(\cdot)$ is not homogeneous of degree n .
29. Find the third-degree Taylor approximation to $(x, y) = (x + y)^2$ at $(0, 0)$ and $(1, 1)$.
30. Find the third-degree Taylor approximation to $f(x, y, z) = x^3 y^2 z$ at $(-1, 0, 1)$.
31. Consider the function $g(x, y) = x^2 + y^2$.
- Graph $\{(x, y) : g(x, y) = 8 \text{ and } (x, y) \geq 0\}$.
 - Find an equation of a line tangent to the curve that you drew in part (a) at the point $(x, y) = (2, 2)$.
 - Let $f(t) = (t, t^2)$. Use the chain rule to compute all partial derivatives of $f \circ g(\cdot)$ at $(x, y) = (2, 2)$.

32. Show that $f(x, y) = (x/[x^2+y^2], y/[x^2+y^2])$ is locally invertible in a neighborhood of every point except the origin. Compute the inverse function explicitly.
33. Show that the following mappings are locally invertible everywhere.
- (a) $f(x, y) = (e^x + e^y, e^x - e^y)$.
 - (b) $f(x, y) = (x + y, x - 2y)$.
 - (c) $f(x, y) = (e \cos y, e \sin y)$.

34. Decide whether it is possible to solve the equations $xu^2 + yzv + x^2z = 3$ and $xyv^3 + 2zu - u^2v^2 = 2$ for (u, v) near $(1, 1)$ as a function of (x, y, z) near $(1, 1, 1)$.
35. Suppose that the equation $f(x, y, z) = 0$ can be solved for each of the three variables as a differentiable function of the other two. Denote by $X(y, z)$ the solution to $f(X(y, z), y, z) = 0$. Similarly define $Y(x, z)$ and $Z(x, y)$. Show that

$$\frac{\partial X}{\partial y} \frac{\partial Y}{\partial z} \frac{\partial Z}{\partial x} = -1$$

36. Let $w = f(x, y, z)$ and $z = g(x, y)$ where $f(\cdot)$ and $g(\cdot)$ are continuously differentiable. Find the mistake in the following argument.

By the chain rule it follows that

$$\frac{\partial w}{\partial x} = \left(\frac{\partial w}{\partial x}\right)\left(\frac{\partial x}{\partial x}\right) + \left(\frac{\partial w}{\partial y}\right)\left(\frac{\partial y}{\partial x}\right) + \left(\frac{\partial w}{\partial z}\right)\left(\frac{\partial z}{\partial x}\right).$$

Therefore, since $\frac{\partial x}{\partial x} = 1$ and $\frac{\partial y}{\partial x} = 0$,

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial x} + \left(\frac{\partial w}{\partial z}\right)\left(\frac{\partial z}{\partial x}\right)$$

and, consequently,

$$\left(\frac{\partial w}{\partial z}\right)\left(\frac{\partial z}{\partial x}\right) = 0.$$

However, when $w = x + y + z$ and $z = x + y$, direction computation confirms that $\frac{\partial w}{\partial z} = \frac{\partial z}{\partial x} = 1$.

37. Consider the function $f(x, y) = x^3y + y^2$ defined on \mathbb{R}^2 .
- (a) Graph $\{(x, y) : f(x, y) = 0\}$.
 - (b) Find an equation of the hyperplane tangent to the graph of $f(x, y) = z$ at the point $(x, y, z) = (0, 2, 4)$.

- (c) Find the equation of some line that lies in the tangent hyperplane that you found in part b.
- (d) Compute the directional derivative of the function $f(\cdot)$ in the direction $v = (3/5, 4/5)$.
38. If $F(x, y, z) = f[u(x, y, z), v(x, y, z)]$ evaluate $D_1F = \frac{\partial F}{\partial x}$ at the point (x_0, y_0, z_0) if $u(x, y, z) = (x^2 - 4y)e^x$, $v(x, y, z) = xyz$, $f(u, v) = \sin(u + v)$, and $(x_0, y_0, z_0) = (2, 1, 0)$.
39. Consider the pair of equations $x^2 + yz = 10$ and $xy + z = 2$.
- (a) Show that it is possible to solve these equations for x and y in terms of z near the point $(x^*, y^*, z^*) = (0, 5, 2)$.
- (b) Compute the derivatives of the implicit functions evaluated at (x^*, y^*, z^*) .
- (c) It is not possible to solve the equations for x and y in terms of z near the point $(x^*, y^*, z^*) = (1, 1, 0)$. Explain why not.

Compute the following indefinite integrals.

40. (a) $\int [(x + 1)dx]$.
- (b) $\int \frac{\log x}{x} dx$.
- (c) $\int xe^x dx$.
- (d) $\int \frac{1}{x} dx$.
- (e) $\int \frac{x^2}{x^3 + 1} dx$.
41. Let $f(x) = [x]$ be the greatest integer function. That is, let $f(x) = n$, where n is the greatest integer less than or equal to x .
- (a) Identify the points at which $f(\cdot)$ is continuous.
- (b) Identify the points at which $f(\cdot)$ is differentiable.
- (c) Compute $\int_0^5 f(x) dx$.
42. Compute $\int \int_A f(x, y) dx dy$ for the following sets A and $f(\cdot)$.
- (a) $A = \{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq 1\}$ and $f(x, y) = xe^y$. b.
- (b) $A = \{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq x\}$ and $f(x, y) = \alpha x + \beta y$.
- (c) $A = \{(x, y) : 0 \leq x \leq 1, x - x^2 \leq y \leq 2x\}$ and $f(x, y) = 10$.
- (d) $A = \{(x, y) : 0 \leq x \leq 1, 1 \leq x + y\}$ and $f(x, y) = x/(y^2)$.
- (e) $A = \{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq x^2\}$ and $f(x, y) = \exp(x + y)$.

Integrate to find the area of the following regions.

43. (a) The circle $x^2 + y^2 \leq 9$.
- (b) The region bounded by the curve $x^3 = y^2$ and the line $x = 1$.

- (c) The region with boundaries $xy = 1$, $xy = 4$, $xy^2 = 1$, and $xy^2 = 8$.
- (d) A triangle with vertices $(0, 1)$, $(1, 0)$, and $(4, 4)$.
44. Compute $f'(x)$ for the following functions.
- (a) $f(x) = \log^2 x$.
- (b) $f(x) = x^2 \cos x$.
- (c) $f(x) = \int_0^x \{\cos y \log(y^2 + 1) \exp[\sin(y - 1)]\} dy$.
- (d) $f(x) = \int_{1-x}^{2x^2+1} [2xy + x^2(2y - 1)] dy$.
- (e) $f(x) = \exp 3$.
- (f) $f(x) = \int_0^1 (x + 1) dy$.
- (g) $f(x) = x^3$.
- (h) $f(x) = 3^x$.
45. Find the general solutions to the following ordinary differential equations.
- (a) $\dot{x} = x/2$.
- (b) $\alpha \dot{x} = \beta$ for $\alpha \neq 0$.
- (c) $\dot{x} + tx = t^3 x^3$.
- (d) $x \dot{x} = t$.
- (e) $t \dot{x} = 2x$.
- (f) $\dot{x} = x(1 - t)$.
46. Specify an initial condition for each of the equations above and solve the associated initial value problem.