Economics 200C: Problem Set III
Due: May 5, 2010

1. Let G' be obtained from G by adding a strictly dominated strategy for Player 1. Can the infinitely repeated version of G' with discount factor δ have a strictly larger set of subgame perfect equilibria than the infinitely repeated version of G with discount factor δ.

   Yes. Adding a strictly dominated strategy for Player 1 can decrease Player 2's minmax, increasing the set of subgame perfect equilibrium payoffs.

	Left	Right
Up	$4, -10$	$-1, -10$
Down	$5, -1$	$0, 0$

   Suppose that the original game G consists of the bottom row of the game above (and both columns). The only equilibrium payoff is $(0, 0)$. Adding the top row allows equilibria in which Column receives a negative payoff.

2. Suppose that two players play the Battle of the Sexes Game:

	Left	Right
Up	$4, 1$	$0, 0$
Down	$0, 0$	$1, 4$

   thirty times with no discounting ($\delta = 1$). Show that there is a subgame perfect equilibrium in which the first period outcome is $(Down, Left)$.

   Strategy for Player 1: start with Down. If (Down,Left) is first period outcome continue by always playing Up. Otherwise, continue by mixing $(.8, .2)$. Strategy for Player 2: start with Left. If (Down,Left) is the outcome, continue by playing Down. Otherwise, continue by mixing $(.2, .8)$. Player 1 earns 116 on the equilibrium. Player 2 earns 29. The only possible profitable deviation is in the first period (afterwards agents always play the same stage-game Nash equilibrium so there is nothing to gain by deviating). For both players a first-period deviation yields $4 + 29 \times .8 < 28$.

3. How many pure strategies does Player 1 have in the four-time repeated battle of the sexes? (Express the number as a power of two.)

   Number of histories in period t: 4^{t-1}. Number of information sets: $1 + 4 + 4^2 + 4^3 = 85$. Number of actions at each information set: 2. Number of strategies: 2^{85}.

4. Suppose that G is a two player game in which Player 1 has k_1 pure strategies and Player 2 has k_2 pure strategies. Write down a formula for the

number of pure strategies Player 1 has in the n time repeated version of G.

Number of actions each period: k_1

Number of t period histories: $(k_1 k_2)^t$

Answer: $\sum_{t=0}^{n-1} k_1(k_1k_2)^t = k_1[(k_1k_2)^n - 1]/(k_1k_2 - 1)$

5. Write down a game G in which the max min values for each player are strictly lower than any Nash equilibrium payoff of G.

	Left	Right
Up	$4, 4$	$-10, -10$
Down	$-10, -10$	$-10, -10$

Minmax is -10 for both players. Unique NE (4,4). The payoffs are not generic, which causes the feasible set to be one dimensional. The assumptions of the folk theorem (and, in fact, the conclusion fails), but you can easily construct an example in which there are equilibria to the repeated game with average payoff strictly less than any stage-game NE payoffs.

6. Consider the prisoner's dilemma:

	Left	Right
Up	$4, 4$	$0, 5$
Down	$5, 0$	$1, 1$

The tit-for-tat strategy for Player i is the strategies that specifies that Player i cooperates in the first period and any period $t > 1$, Player i plays what Player j played in period $t-1$ ($j \neq i$). The questions below refer to the infinitely repeated prisoner's dilemma.

(a) Suppose that Player 1 plays tit-for-tat. For what values of δ is it a best response for Player 2 to play tit-for-tat?

Player 2 receives 4 for following the equilibrium. The set of potentially attractive deviations involve picking a period to deviate and then specifying continuation behavior. There are essentially two possibilities: Deviate today and and continue to deviate in the next period or deviate today and cooperate tomorrow. The first deviation is unattractive if $4 \geq (1-\delta)5 + \delta$; the second is unattractive if $4 \geq 5/(1+\delta)$.

(b) For what values of δ is it a subgame perfect equilibrium for both players to play tit-for-tat. We look at Player 2 (Player 1 has a symmetric problem). Now we need to check for deviations in all subgames. There are four kinds:

i. ones that follow (C, C),
If Player 2 makes a one-time deviation on the equilibrium path the outcome will switch back and forth between (C, D) and (D, C). By the one-time deviation principle, it is sufficient to check that this is not attractive. The payoff for deviating is $5/(1 + \delta)$ which is less than or equal to 4 if $\delta \geq .25$.

ii. ones that follow (D, C), where Player 1 starts by cooperating and Player 2 is supposed to deviate (according to tit-for-tat); if he does, he earns $5/(1 + \delta)$; if he doesn't he earns 4, so the condition is the reverse of the previous part: you need $\delta \leq .25$.

iii. ones that follow (C, D) in which Player 1 starts with D and Player 2 receives $5\delta/(1 + \delta)$ by playing tit-for-tat and 1 for deviation (playing D) in the first period of the subgame. This deviation is not attractive if $\delta \geq .25$.

iv. ones that follow (D, D) in which the equilibrium yields payoff 1 and cooperation yields $5\delta/(1 + \delta)$.

So the answer is that you must have $\delta = .25$.

(c) Suppose Player 1 plays tit-for-tat and Player 2 cooperates in periods divisible by three and defects in all other periods (independent of history). What are the payoffs? Does either player best respond to the other (your answer may depend on δ)?

The outcome cycles through (C, D), (D, D), and (D, C). Player 1 receives average payoff $(1 - \delta)(1 + 5\delta)\delta/(1 - \delta^3)$; Player 2 receives $(1 - \delta)(5 + \delta)/(1 - \delta^3)$

Player 1 is not best responding. He would do better to deviate in every period. If Player 2 cooperates in the second period (and then goes pack to the original strategy) the original outcome changes in the second and third periods: (D, D) then (D, C) becomes (D, C) then (C, C); the deviation is profitable if $1 < \delta 4$ or $\delta > .25$. (The same inequality holds for first-period deviations.) If Player 2 deviates in the third period then (D, C) followed by (C, D) (in third and fourth periods) becomes (D, D) followed by (D, D). So the deviation is attractive if $\delta < .25$. If $\delta = .25$, then Player 2 is playing a best response.