Econ 200C, Spring 2012
Problem Set #3
Due: Never, but fair game for exams

1. Consider a signaling model in which firms can observe workers' education levels, but not their productivities, when they decide which workers to hire, and workers, who know their own productivities, choose their education levels. Assume that the output of a worker of type n who has had y years of education is $S(n, y) = ny^a$, where $0 < a < 1$, and that the cost of y years of education to a worker of type n is $C(n, y) = y/n$. Let $w(y)$ be the wage offered in equilibrium to a worker with y years of education. Assume that $S(\cdot)$, $C(\cdot)$, and $w(\cdot)$ are differentiable.

   (a) Assuming that each worker selects his education level to maximize the difference between his wage and the cost of his education, write the first-order condition that determines how much education a worker of type n chooses to acquire.

   The worker maximizes: $w(y) - y^2/n$ by choosing y. The first-order condition is $w'(y) = 1/n$. (This assumes that $w(\cdot)$ is differentiable.)

   (b) Briefly explain why, in a separating equilibrium, the wage offered a worker with y years of education equals the output of the type of worker who chooses that level of education. (The problem essentially asks you to repeat/recall familiar arguments from class.)

   If wage is higher than output, then firms make negative profit if the wage offer is accepted. They would do better offering a lower wage. If wage is lower, then some firm can increase its wage offer and thereby increase profit.

   (c) Using your answers to the first two parts, derive an equation that describes the equilibrium relationship between a worker's wage and her chosen level of education in a separating equilibrium.

   We know that a worker with education y will be paid $S(n, y)$. We need notation for the type of worker that gets education y. In a separating equilibrium, there will be just one, so we have a function $N(\cdot)$, where $n = N(y)$ is the type of a worker with education y. Hence $w(y) = S(N(y), y) = N(y)y^a$ and the first-order condition is $N'(y)y^a + aN(y)y^{a-1} = 1/N(y)$.

   (d) Assuming that $a = 0$ (so education is not productive), determine the level of education chosen by a worker of type n in an equilibrium with parameter K. If n is continuously distributed on $[1, 2]$, for what values of K will your solution from (c) satisfy the nonnegativity constraint on education?

   The problem asks to solve the differential equation $N'(y) = 1/N(y)$. It follows that $N^2(y) = 2y + K$ and therefore the education level of a type n worker, $Y(n)$ is the inverse function $Y(n) = (n^2 - K)/2$ and we need $K \leq 1$ for non-negativity.

(e) Indicate which workers would gain, when $a = 0$ (education is abolished).

Without education, workers receive average type, $\bar{n}$ so the question is: For what values of n is $\bar{n} > n - (n^2 - K)/(2n) = (K/n + n)/2$. In particular, if n is uniformally distributed and $K = 1$ ($K = 1$ is the "efficient" separating equilibrium), then the types that gain satisfy: $3 > 1/n + n$ and everyone gains.

2. Sue was in a car accident. Roger, the other driver, was responsible. Sue knows the extent of her injuries, t. Roger knows only that t is equal to 10 with probability q and 1 with probability $1 - q$.

Suppose that Roger has the opportunity to make a take-it-or-leave it settlement offer. If Roger offers s and Sue accepts, then Roger earns $-s$ and Sue earns s. If Roger offers s and Sue rejects, then the case goes to court. Roger earns $-t - c$ and Sue earns t where $c > 0$ are court costs.

(a) Find the pure strategy equilibria of this game. Does the solution concept matter? Distinguish between Nash, subgame perfect, and perfect Bayesian equilibria.

Sue's strategy which s to accept (type contingent). In a subgame perfect equilibrium, she'll accept offers greater than t and reject lower offers. Roger has essentially two choices, he can offer 10. Sue will accept with probability one and Roger will get -10. Alternatively, he can offer 1 and receive $-1(1-q) - (10+c)q$. Offering 10 is better if $10 < (1-q) + (10+c)q$ or $c > 9(1-q)/q$. That is, Roger settles all of the time if the cost of going to court is high or the probability that Sue was badly damaged is high. There is no difference between SGP and PBE for this game.

This discussion describes SGPE. There are more NE. For example, if Sue rejects any s other than $s^* > 10$, then Roger's payoff from s^* is $-s^*$ and from any other offer is $-(1-q) - 10q - c$. Provided that $s^* \leq (1-q) + 10q + c$ this will be an equilibrium.

Now assume that Sue has the opportunity to make a take-it-or-leave-it settlement demand. If Sue offers x, and Roger accepts, then Sue earns x and Roger $-x$. If Sue offers x and Roger rejects, then Sue earns t and Roger $-t - c$. Assume that Sue can only offer $10 + c$ or $1 + c$ and that Sue's (type-contingent) strategy is pure but that Roger can randomize in response to a high demand.

(b) Find all equilibria of this game. Distinguish between Nash, subgame perfect, and perfect Bayesian equilibria. Does a pooling equilibrium exist? Does a separating equilibrium exist?

Now the informed player goes first. Given the constraints on the strategy space, the possible strategies for Sue are always ask $1 + c$; always ask $10 + c$; separate, asking $t + c$. In the first case, Roger will accept $1 + c$ and (specifying his full strategy) accept $10 + c$ with probability α. This will be an equilibrium if Sue does not want to deviate when $t = 10$. If $t = 10$ she earns $10 + c\alpha$ (because if Roger accepts, Sue gets $10 + c$ and she gets 10 otherwise). Hence we have an equilibrium in which Sue pools at $1 + c$ provided that $1 + c \geq 10 + c\alpha$ or $c \geq 9/(1 - \alpha)$. Of course this is only possible for some $\alpha \in [0, 1]$ if $c \geq 9$. If Sue always asks $10 + c$, then Roger will reject with probability one because he will pay $(1 - q) + 10q + c$ if he rejects. This can be an equilibrium if Roger rejects the lower offer. That is possible in a Nash equilibrium and (there are no proper subgames) in a subgame perfect equilibrium. It is also possible in a perfect Bayesian equilibrium (Roger believes that the unexpected low demand comes from $t = 1$ with probability 1 and always rejects it, which is ok, since with those beliefs he is indifferent). For the record, the pooling at $10 + c$ outcome is not trembling hand perfect because it is weakly dominated for Roger to reject the low demand. Finally, if Sue separates, Roger is indifferent. Suppose that he always accepts the low demand and accepts the high one with probability α. Roger is best responding. Sue is best responding when $t = 1$ provided that $1+c \geq \alpha(10+c)+(1-\alpha)$ or $\alpha \leq c/(1+c)$ and when $t = 10$ provided that $(10+c)\alpha \geq 1+c$. There could be separating equilibria in which the lower demand is rejected with positive probability. This would place more restrictive assumptions on α.

3. David has n items, $n \in \{0, \ldots, N\}$. He can make a take-it-or-leave-it offer to Nageeb. Call the offer p. If Nageeb accepts the offer, then David earns p and Nageeb earns $V_n - p$. Assume that $V_n > V_{n-1}$.

   (a) Find all pure-strategy Nash equilibria of the game in which nature first selects n ($q_n > 0$ is the probability that David is given n items for $n \in \{0, \ldots, N\}$); David learns n; David offers p; Nageeb accepts or rejects p (Nageeb knows the probability distribution q_i, but not n).

   Suppose that Nageeb accepts some p. David will then ask the highest price Nageeb accepts (if no such price exists, then there is no equilibrium with this acceptance set for Nageeb). On the other hand, Nageeb will only accept p if $p \leq \sum V_n q_n = \bar{V}$, the expected value. For any set that has a maximum element less than or equal to $\bar{V}$ there is an equilibrium in which David asks the upper bound (for all n) and Nageeb accepts all bids in the set and rejects all others. This describes all pure strategy NE.

   (b) Find all pure-strategy subgame perfect Nash equilibria of the game.

   There are no proper subgames (so the answer does not change).

   (c) Suppose now that David can make a statement (an element from a set M) prior to making a price offer. Payoffs are exactly as before. Answer the first two parts.

David will send the message that leads to the highest payoff in the continuation. That means that David will be indifferent over all messages that he sends in equilibrium. In NE different types can send different messages in equilibrium provided that in any case the acceptance sets have the same highest element.

(d) Modify the previous question assuming that M consists of all non-empty subsets of $\{0, 1, \ldots, N\}$ and that if David observes n, he can only signal using elements $m \in M$ such that $n \in m$. (That is, David's information is verifiable: he can hide information by sending a "vague" message (m consisting of more than one point), but he cannot lie $n \notin m$).

There is a pooling equilibrium in which all types use the message $m = \{0, 1, \ldots, N\}$ and acceptance behavior is as in the first part. Here it is interesting to study refinements. Given a message m in which the lowest element of m is n, Nageeb must believe that there are at least n items and must accept any price less than V_n. Hence each type can guarantee V_n by using $m = \{n\}$. If this is the case, no type can be pooled with a lower type. Hence the equilibrium is completely revealing. For each m, Nageeb must believe that the message is sent by the lowest type in m, call it $n(m)$ and will accept prices no higher than $V_{n(m)}$. Each type uses m that contains no lower types. Intuitively, David tries to exaggerate and Nageeb is skeptical (and believes "the worst").