Solution to Matching Problems

Exercise 1 Construct an example in which there is more than one stable matching. (You only need two boys and two girls to do this.)

Solution 1 Suppose the preferences are:
Alan: Megan $\succ$ Melissa.
Ron: Melissa $\succ$ Megan.
Melissa: Alan $\succ$ Ron.
Megan: Ron $\succ$ Alan.
So the match Alan-Megan, Ron-Melissa is stable. (This is the favorite of the boys.) The match Alan-Melissa, Ron-Megan is also stable. (This is the favorite of the girls.)

Exercise 2 Suppose that the boys all have different favorite girls. How many steps does it take for the algorithm to converge?

Solution 2 One. They all propose to a different girl.

Exercise 3 Suppose that the boys have identical preferences. How many steps does it take for the algorithm to converge?

Solution 3 At the first round, all boys propose to the same girl. She "dates" her favorite. At the second round, all but that favorite one boy proposes to the second-best girl. She "dates" her favorite. The process continues for N rounds until one guy is left to propose to the least desirable girl.

Exercise 4 Suppose preferences are given by the following tables:

BOY	1	2	3	4	5
Adam	Beth	Amy	Diane	Ellen	Cara
Bill	Diane	Beth	Amy	Cara	Ellen
Carl	Beth	Ellen	Cara	Diane	Amy
Dan	Amy	Diane	Cara	Beth	Ellen
Eric	Beth	Diane	Amy	Ellen	Cara

Boys' Preferences

GIRL	1	2	3	4	5
Amy	Eric	Adam	Bill	Dan	Carl
Beth	Carl	Bill	Dan	Adam	Eric
Cara	Bill	Carl	Dan	Eric	Adam
Diane	Adam	Eric	Dan	Carl	Bill
Ellen	Dan	Bill	Eric	Carl	Adam

Girls' Preferences

Find a stable matching using the Gale-Shapley algorithm with boys making proposals. Find a stable matching using the Gale-Shapley algorithm with girls making proposals.

Solution 4 If the boys make proposals, then you get:

Round 1:

GIRL	Date	Rejected this round
Amy	Dan	
Beth	Carl	Adam, Eric
Cara		
Diane	Bill	
Ellen		

Round 2:

GIRL	Date	Rejected this round
Amy	Adam	Dan
Beth	Carl	
Cara		
Diane	Eric	Bill
Ellen		

Round 3:

GIRL	Date	Rejected this round
Amy	Adam	
Beth	Carl	Bill
Cara		
Diane	Eric	Dan
Ellen		

Round 4:

GIRL	Date	Rejected this round
Amy	Adam	
Beth	Carl	
Cara	Dan	Bill
Diane	Eric	
Ellen		

Round 5:

GIRL	Date	Rejected this round
Amy	Adam	Bill
Beth	Carl	
Cara	Dan	
Diane	Eric	
Ellen		

Round 6:

GIRL	Date	Rejected this round
Amy	Adam	
Beth	Carl	
Cara	Bill	Dan
Diane	Eric	
Ellen		

Round 7:

GIRL	Date	Rejected this round
Amy	Adam	
Beth	Carl	Dan
Cara	Bill	
Diane	Eric	
Ellen		

Round 8:

GIRL	Date	Rejected this round
Amy	Adam	
Beth	Carl	
Cara	Bill	
Diane	Eric	
Ellen	Dan	

This is the boy-optimal stable matching.
If girls make proposals, then we have:

Round 1:

Boy	Date	Rejected this round
Adam	Diane	
Bill	Cara	
Carl	Beth	
Dan	Ellen	
Eric	Amy	

This is the girl-optimal stable matching.

Exercise 5 This exercise shows that stable matchings need not exist if there are not "two sides." Consider the following "roommate" problem. There are four people, Pat, Chris, Dana, and Leslie. They must pair off (each pair will share a two-bed suite). Each has preferences over which of the others they would prefer to have as a roommate. The preferences are:

Leslie: Pat $\succ$ Chris $\succ$ Dana
Chris: Leslie $\succ$ Pat $\succ$ Dana
Pat: Chris $\succ$ Leslie $\succ$ Dana
Dana: Chris $\succ$ Leslie $\succ$ Pat

Show that no stable matching exists. (That is, no matter who you put together, they will always be two potential roommates who are not matched, but prefer each other to their current roommate.)

Solution 5 Someone must be matched with Dana. Whoever that is would prefer either of the other two to Dana. One the other hand, one of the others will think that Dana's partner is the best roommate. (Concretely, if the pairings are Dana-Chris and Leslie-Pat are paired, then Pat and Chris would prefer to be with each other than their mate. The same kind of argument would work whoever paired with Dana.)