Econ 172A, Fall 2012: Midterm Solutions

Comments

1. 140 possible, high 134, low 9, median 97, standard deviation 22.
2. Please read the solutions to evaluate your performance. (I would appreciate corrections too.)
3. If you do have questions, please come to me.
4. If you think that your grade is inaccurate, please follow the posted instructions.

Grading Notes

1. There are 5 parts. Maximum for each part: 6, 6, 18, 10, 1. Minimal deductions for missing non-negativity in (a) or (b). In (c) full credit required identifying the right system of equations to solve, explaining why it is the right system, and solving it. In (d) many people lost 6 points for failing to describe correctly why they have (have not) found a solution.
2. There are 8 parts on the page and 11 parts on the grid. Give 2 points for each of the 8 and 3 for each of the 11 for a total of 49. No justifications necessary.
3. There are 8 parts. 10 points for parts (b) and (h) and 5 each for the others. The key on this question is to realize that each pill is just a mixture of the two Chemicals. The problem is how to "bundle" the ingredients together.

Form A Solutions

1. (a) The dual is: Find y_1, y_2, and y_3 to solve **D**:

$$\begin{array}{lrcrcrcr} \min & 10y_1 & + & 20y_2 & + & 5x_3 & & \\ \text{subject to} & y_1 & + & y_2 & + & y_3 & \geq & 10 \\ & 2y_1 & & & + & y_3 & \geq & 2 \\ & 3y_1 & - & y_2 & & & \geq & 0 \\ & 4y_1 & + & 2y_2 & - & y_3 & \geq & 5 \\ & & & y & \geq & 0 & & \end{array}$$

   (b) Check feasibility: First and third constraints bind. Second constraint has slack. Variables nonnegative, with x_1 and x_4 strictly positive. Conclude that x is feasible.

   (c) If x solves **P**, then it must be that first and fourth dual constraints bind (because x_1 and x_4 are positive) and $y_2 = 0$ (because second constraint in primal does not bind). Hence we must solve:

   $y_1 + y_3 = 10$ and $4y_1 - y_3 = 5$. These equations have the unique solution $y_1 = 3$ and $y_3 = 7$. So CS says that the candidate solution to **D** is $y = (3, 0, 7)$.

   (d) Yes. We need to check that the candidate solution satisfies the omitted constraints: $y \geq 0$, $2y_1 + y_3 \geq 2$ and $3y_1 - y_2 \geq 0$. Plug the values of y into these constraints and conclude that y is in fact feasible for **D**. Since the value x gives in the primal (65) is equal to the value y gives in the dual, they are solutions.

   (e) Yes. We computed it above.

2. (a) 4

   (b) 3

   (c) $15x_1 + 6x_2 + 9x_3 + 2x_4$

   (d) $20y_1 + 70y_2 + 24y_3$

   (e) 132

   (f) 132

   (g) $x_1 = 4, x_2 = 12, x_3 = x_4 = 0$

   (h) $y_1 = 3, y_2 = 0, y_3 = 3$

3. (a) Constraint (3) states that the total amount of pills produced (in ounces) is less than or equal to the total number of ounces of chemical produced. The left-hand side is the total amount of pills produced. Process P produces 3 ounces of each chemical (a total of 6 ounces) for each hour of operation. Process Q produces 3 ounces of Chemical A and one ounce of Chemical B (a total of 4 ounces) for each hour of operation. Hence the right-hand side is the total amount of chemicals produced.

   (b) It is feasible to set all variables equal to zero. The problem is bounded because there are limits to the amount of Chemical A and B (and hence Pill 1 and 2) that can be produced. Concretely, (1) (and non-negativity) guarantees that $z_P, z_Q \leq L$ and so (3) bounds total pill production.

   (c) Yes. Suppose you have a production plan involving x_i as output of Pill i. Form a new plan in which you produce $x_1' = 0$ of Pill 1 and $x_2' = x_1 + x_2$ of Pill 2. This production plan is feasible (satisfies all of the constraints) and earns more profits (when $\pi_2 > \pi_1$). The simple intuition is that Pill 1 is the "harder" pill to product because it requires more of Chemical 1. If the harder pill is less profitable, then there is no need to produce it. (Constraint (4) is easier to satisfy if you shift production from Pill 1 to Pill 2.)

(d) No. Now the harder pill is more profitable.

(e) The solution does not change. The value doubles.

(f) I do not know what happens to the solution. (In fact, one can be confident that production of the "easier" pill does not go down because it is relatively more profitable, but I didn't expect you to say this.) The value goes up by at least the amount of total production $(x_1^* + x_2^*)$.

(g) If Constraint (1) is not binding.

(h) $y_{1A} \geq .65x_1$ states that the amount of Chemical A in Pill 1 is at least 65% of the amount of Pill 1 produced. Also, $3z_P + 3z_Q$ is the total amount of Chemical A produced. At least 55% of Pill 2 is Chemical A, so $3z_P + 3z_Q - .55x_2 \geq y_{1A}$.

Form B Solutions

1. (a) The dual is: Find y_1, y_2, and y_3 to solve **D**:

$$\begin{array}{lccccccc}
\min & 20y_1 & + & 20y_2 & + & 5x_3 & & \\
\text{subject to} & 2y_1 & + & y_2 & + & y_3 & \geq & 10 \\
 & 4y_1 & & & + & y_3 & \geq & 20 \\
 & 6y_1 & - & y_2 & & & \geq & 0 \\
 & 8y_1 & + & 2y_2 & - & y_3 & \geq & 5 \\
 & & & y & \geq & 0 & &
\end{array}$$

   (b) Check feasibility: First and third constraints bind. Second constraint has slack. Variables nonnegative, with x_1 and x_4 strictly positive. Conclude that x is feasible.

   (c) If x solves **P**, then it must be that first and fourth dual constraints bind (because x_1 and x_4 are positive) and $y_2 = 0$ (because second constraint in primal does not bind). Hence we must solve:

   $2y_1 + y_3 = 10$ and $8y_1 - y_3 = 5$. These equations have the unique solution $y_1 = 1.5$ and $y_3 = 7$. So CS says that the candidate solution to **D** is $y = (1.5, 0, 7)$.

   (d) No. We need to check that the candidate solution satisfies the omitted constraints: $y \geq 0$, $4y_1 + y_3 \geq 20$ and $6y_1 - y_2 \geq 0$. Plug the values of y into these constraints and conclude that y is not feasible for **D** (because $4y_1 + y_3 = 13 < 20$. Hence neither the given x nor the computed y can be solutions.

   (e) No, by the preceding discussion.

2. (a) 4

   (b) 3

   (c) $30x_1 + 12x_2 + 18x_3 + 4x_4$

   (d) $20y_1 + 70y_2 + 24y_3$

   (e) 264

   (f) 264

   (g) $x_1 = 4, x_2 = 12, x_3 = x_4 = 0$

   (h) $y_1 = 6, y_2 = 0, y_3 = 6$

3. (a) Constraint (3) states that the total amount of pills produced (in ounces) is less than or equal to the total number of ounces of chemical produced. The left-hand side is the total amount of pills produced. Process P produces 3 ounces of each chemical (a total of 6 ounces) for each hour of operation. Process Q produces 3 ounces of Chemical A and one ounce of Chemical B (a total of 4 ounces) for each hour of operation. Hence the right-hand side is the total amount of chemicals produced.

   (b) It is feasible to set all variables equal to zero. The problem is bounded because there are limits to the amount of Chemical A and B (and hence Pill 1 and 2) that can be produced. Concretely, (1) (and non-negativity) guarantees that $z_P, z_Q < L$ and so (3) bounds total pill production.

   (c) No. Now the harder pill is more profitable. (See next answer.)

   (d) Yes. Suppose you have a production plan involving x_i as output of Pill i. Form a new plan in which you produce $x_2' = 0$ of Pill 2 and $x_1' = x_1 + x_2$ of Pill 1. This production plan is feasible (satisfies all of the constraints) and earns more profits (when $\pi_1 > \pi_2$). The simple intuition is that Pill 2 is the "harder" pill to product because it requires more of Chemical 1. If the harder pill is less profitable, then there is no need to produce it. (Constraint (4) is easier to satisfy if you shift production from Pill 2 to Pill 1.)

(e) The solution does not change. The value doubles.

(f) I do not know what happens to the solution. (In fact, one can be confident that production of the "easier" pill does not go down because it is relatively more profitable, but I didn't expect you to say this.) The value goes up by at least the amount of total production $(x_1^* + x_2^*)$.

(g) If Constraint (1) is not binding.

(h) $y_{1A} \geq .55x_1$ states that the amount of Chemical A in Pill 1 is at least 55% of the amount of Pill 1 produced. Also, $3z_P + 3z_Q$ is the total amount of Chemical A produced. At least 75% of Pill 2 is Chemical A, so $3z_P + 3z_Q - .75x_2 \geq y_{1A}$.

Form C Solutions

1. (a) The dual is: Find y_1, y_2, and y_3 to solve **D**:

$$\begin{array}{rrcrcrcr}
\text{min} & 5y_1 & + & 10y_2 & + & 20x_3 & & \\
\text{subject to} & 2y_1 & + & 2y_2 & + & 2y_3 & \geq & 20 \\
 & y_1 & + & 2y_2 & & & \geq & 2 \\
 & -y_1 & + & 4y_2 & + & 2y_3 & \geq & 5 \\
 & & & 3y_2 & - & y_3 & \geq & 0 \\
 & & & y & \geq & 0 & &
\end{array}$$

   (b) Check feasibility: First and second constraints bind. Third constraint has slack. Variables nonnegative, with x_1 and x_3 strictly positive. Conclude that x is feasible.

   (c) If x solves **P**, then it must be that first and third dual constraints bind (because x_1 and x_3 are positive) and $y_3 = 0$ (because third constraint in primal does not bind). Hence we must solve:

   $2y_1 + 2y_2 = 20$ and $-y_1 + 4y_2 = 5$. These equations have the unique solution $y_1 = 7$ and $y_2 = 3$. So CS says that the candidate solution to **D** is $y = (7, 3, 0)$.

   (d) We need to check that the candidate solution satisfies the omitted constraints: $y \geq 0$, $y_1 + 2y_2 \geq 2$ and $3y_2 \geq 0$. Plug the values of y into these constraints and conclude that y is feasible for **D**.

   (e) Yes. Computed above.

2. (a) 4
   (b) 3
   (c) $25x_1 + 24x_2 + 18x_3 + 4x_4$
   (d) $20y_1 + 20y_2 + 24y_3$
   (e) 244
   (f) 244
   (g) $x_1 = 4, x_2 = 6, x_0 = x_4 = 0$
   (h) $y_1 = 11, y_2 = 0, y_3 = 1$

3. (a) The right-hand side is the total amount of Chemical A produced. The left hand side is the minimum amount of Chemical A needed to produce the Pills. So the constraint says that there is enough Chemical A available to satisfy the production requirements.

   (b) It is feasible to set all variables equal to zero. The problem is bounded because there are limits to the amount of Chemical A and B (and hence Pill 1 and 2) that can be produced. Concretely, (1) (and non-negativity) guarantees that $z_P, z_Q < L$ and so (4) bounds total pill production.

   (c) Yes. Suppose you have a production plan involving x_i as output of Pill i. Form a new plan in which you produce $x_1' = 0$ of Pill 1 and $x_2' = x_1 + x_2$ of Pill 2. This production plan is feasible (satisfies all of the constraints) and earns more profits (when $\pi_2 > \pi_1$). The simple intuition is that Pill 1 is the "harder" pill to product because it requires more of Chemical 1. If the harder pill is less profitable, then there is no need to produce it. (Constraint (4) is easier to satisfy if you shift production from Pill 1 to Pill 2.)

   (d) No. Now the harder pill is more profitable.

(e) Yes. Suppose you have a production plan involving x_i as output of Pill i. Form a new plan in which you produce $x_2' = 0$ of Pill 2 and $x_1' = x_1 + x_2$ of Pill 1. This production plan is feasible (satisfies all of the constraints) and earns more profits (when $\pi_1 > \pi_2$). The simple intuition is that Pill 2 is the "harder" pill to product because it requires more of Chemical 1. If the harder pill is less profitable, then there is no need to produce it. (Constraint (3) is easier to satisfy if you shift production from Pill 2 to Pill 1.)

(f) No.

(g) The solution does not change. The value doubles.

(h) I do not know what happens to the solution. (In fact, one can be confident that production of the "easier" pill does not go down because it is relatively more profitable, but I didn't expect you to say this.) The value goes up by at least the amount of total production $(x_1^* + x_2^*)$.

(i) If Constraint (1) is not binding.

(j) $y_{1A} \geq .65x_1$ states that the amount of Chemical A in Pill 1 is at least 65% of the amount of Pill 1 produced. Also, $3z_P + 3z_Q$ is the total amount of Chemical A produced. At least 25% of Pill 2 is Chemical A, so $3z_P + 3z_Q - .25x_2 \geq y_{1A}$.