Econ 172A, Fall 2012: Final Examination Solutions (I)

1. The entries in the table below describe the costs associated with an assignment problem.

   There are four people (1, 2, 3, 4) and four jobs (A, B, C, and D). The entry in column j and row i is the cost associated with assigning person i to job j (so, for example, the cost of assigning person 3 to job C is 60). Find the cost-minimizing assignment of worker to job (each worker should do exactly one job; each job should be assigned to exactly one worker).

	A	B	C	D
1	40	9	10	4
2	100	85	102	98
3	80	70	60	50
4	45	10	12	30

   Step 1: From each row, subtract smallest element in the row.

	A	B	C	D
1	36	5	6	0
2	15	0	17	13
3	30	20	10	0
4	35	0	2	20

   Step 2: From each column, subtract smallest element in the column.

	A	B	C	D
1	21	5	4	0
2	0	0	15	13
3	15	20	8	0
4	20	0	0	20

   Step 3: Attempt to find zero cost match and fail. (1 and 3 both must match with D).

   Step 4: Cross out zeros in with three lines. (For example, Row 2 and 4 and Column D.)

   Step 5: Subtract smallest uncrossed number from every entry (4). Add back this amount to each row or column crossed out.

	A	B	C	D
1	17	1	0	0
2	0	0	15	17
3	11	16	4	0
4	20	0	0	24

   Step 6: Attempt to find a zero cost match and succeed. 3 must go with D, so 1 must go with C, so 4 with B, and 2 with A: 2-A; 1-C;3-D;4-B: cost $100 + 10 + 50 + 10 = 170$.

2. Consider a marriage problem in which there are four men, (1, 2, 3, 4), and four woman (A, B, C, and D). The preferences of the men are:

   Suppose preferences are given by the following tables:

   Mens' Preferences

MEN				
1	C	D	B	A
2	A	D	B	C
3	B	A	C	D
4	B	D	A	C

   Womens' Preferences

WOMAN				
A	1	2	3	4
B	1	4	3	2
C	2	4	3	1
D	4	1	2	3

   So, for example, Man 1 prefers Woman C to Woman D, Woman D to Woman B, and Woman B to Woman A.

   (a) Exhibit a stable marriage for this problem. Justify your answer.

   Run algorithm with Men proposing.

   First: 1-C,2-A,3-B,4-B; B rejects 3.

   Second: 1-C,2-A,3-A,4-B; A rejects 3.

   Third: 1-C,2-A,3-C,4-B; C rejects 1.

   Fourth: 1-D,2-A,3-C,4-B; this is a stable marriage because all women have partners.

   (b) Is the stable marriage you found unique? If not, exhibit another stable marriage. If so, explain why it is unique.

   Run the algorithm with Woman proposing.

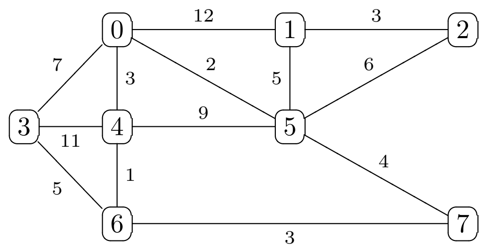
   First: A-1,B-1,C-2,D-4; 1 rejects A.

   Second: A-2,B-1,C-2,D-4; 2 rejects C.

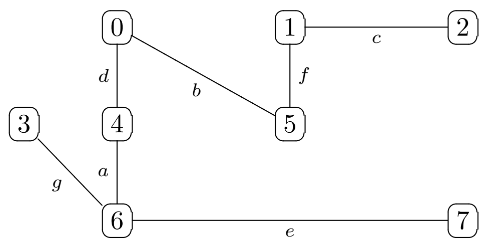
   Third: A-2, B-1, C-4, D-4; 4 rejects C.

   Fourth: A-2, B-1, C-3, D-4. This is a different stable marriage. (So you don't get uniqueness.)

3. Find a minimal spanning tree for the network below (the numbers on the edges are costs):



Draw a minimal spanning tree for the network above here (draw in the edges in the minimum spanning tree):



Briefly describe how you found this tree.

I added the cheapest unused edge at each step provided that adding the edge doesn't create a cycle. (I added edges in numerical order: $a, b, c, \ldots$.) I stopped when I had 7 links. Notice that there were ties (for example, the third through fifth edges added had cost 3 and could have been added in any order) and that I didn't add the edge connecting Node 5 to Node 7 because it would have formed a cycle.

Compute the cost of the spanning tree that you found above:

In the order I added edges:

$$1 + 2 + 3 + 3 + 3 + 5 + 5 = 22$$

4. Consider the network on the next page. (Note: I have included several copies of the network so that you can use it to show your work. Depending upon how you solve the problem, you may need fewer or more networks.) The numbers on the edges are capacities.

   (a) Find a maximum flow (from the source node (s) to the sink node (n)) for this network. You may describe the flow here or on one of the network diagrams on the next page. Please indicate your answer clearly. You must explain how you found the answer. If you properly used the algorithm introduced in class (and you show the steps), then you need no further justification. If you used another method, then you must explain the method and explain how it works.

       Next page.

   (b) Find the value of the maximum flow.

       60

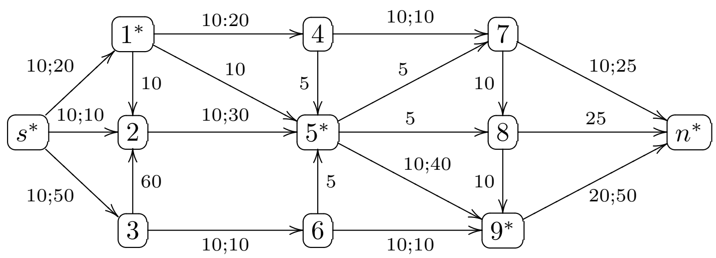
   (c) Find a minimum capacity cut for this network.

       $\{s, 2, 3\}$ and $\{1, 4, 5, 6, 7, 8, 9, n\}$

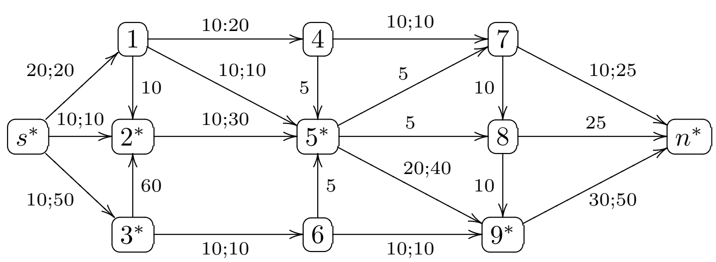
   (d) What is the capacity of the minimum capacity cut? 60.

   (e) Find the capacity of the cut $\{s, 1, 2, 4, 5\}$ and $\{3, 6, 7, 8, 9, n\}$.

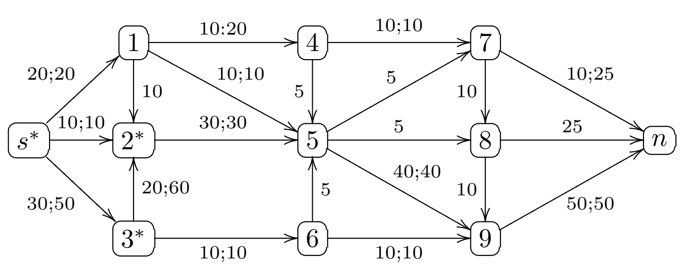
       The capacity of this cut is the sum of the capacities of the links: s to 3; 4 to 7, 5 to 7, 5 to 8 and 5 to 9: $50 + 10 + 5 + 5 + 40 = 110$.



I put in three "easy" paths: sending 10 from s to 1 to 4 to 7 to n; sending 10 from s to 2 to 5 to 9 to n; sending 10 from s to 3 to 6 to 9 to n. I then applied the labeling rule to identify the path s to 1 to 5 to 9 to n. This path can carry ten units, so I add them.



I find the path s to 3 to 2 to 5 to 9 to n, which can carry 20 more units.



This is a solution. The minimum capacity cut has s, 2, and 3 in one set and the rest in the other set. The capacity of this cut is 60, which, of course, is equal to the maximum flow.

5. A drug company manufactures two kinds of sleeping pill: Pill 1 and Pill 2. The pills contain a mixture of two different chemicals, A and B (and no other ingredients). By weight, the Pill 1 must contain at least 65% Chemical A and Pill 2 must contain at least 55% Chemical A. There are two different ways to produce Chemicals A and B. Operating Process P for one hour requires 7 ounces of a raw material and 2 hours of labor time (because it must be supervised by 2 workers). Operating Process Q for one hour requires 5 ounces of a raw material and 3 hours of labor time. Operating Process P for one hour produces three ounces of each chemical. Operating Process Q for one hour produces 3 ounces of Chemical A and one ounce of Chemical B. There are $L > 0$ hours of labor and $R > 0$ ounces of raw material are available. The company earns a profit from Pill i of $\pi_i \geq 0$ dollars per ounce. The firm wishes to find a production plan that maximizes the profit it makes from manufacturing sleeping pills (subject to the constraints above).

   Formulate the problem as a linear programming problem.

   Introduce the variables, z_Q, z_P, x_1, and x_2. z_Q represents the number of hours Process Q is operated. z_P represents the number of hours Process P is operated. x_i represents the number of ounces of Pill i produced ($i = 1$ or 2).

   The problem becomes: Find z_Q, z_P, x_1, and x_2 to solve:

   $\max \pi_1 x_1 + \pi_2 x_2$ subject to:

   (1) $2z_P + 3z_Q \leq L$

   (2) $7z_P + 5z_Q \leq R$

   (3) $x_1 + x_2 \leq 6z_P + 4z_Q$

   (4) $.65x_1 + .55x_2 \leq 3z_P + 3z_Q$

   (5) $x_1, x_2, z_P, z_Q \geq 0$

   Explanation: The objective function is clear. Constraint (1) states that there is sufficient labor to run the processes at level z_P and z_Q. Constraint (2) states that there is sufficient raw material to run the processes at level z_P and z_Q. Constraint (3) says that the quantity of pills is no greater than the total quantity of chemical produced. Constraint (4) says that the quantity of pills requires no more chemical A than the amount of chemical A produced.

6. The table below comes from a computation for a shortest-route problem, using the algorithm that I presented in class. (The table looks for shortest routes that begin at Node 1.) The nodes in the network are connected by directed edges that have non-negative costs. Assume that for all i, $c(i,i) = 0$. If $j > i$, there is no direct path from Node i to Node j. In these cases, $c(j,i) = \infty$.

Iteration/Node	1	2	3	4	5	6
1	0*	10	4**	15	17	7
2	0*	10	4*	13	16	5**
3	0*	10**	4*	13	16	5*
4	0*	10*	4*	12	11**	5*
5	0*	10*	4*	12**	11*	5*

(a) What is the cost of the shortest route from Node 1 to Node 5?
11

(b) What is the shortest route from Node 1 to Node 5?
1 to 2 to 5.

(c) What is the cost of the shortest route from Node 1 to Node 4?
12.

(d) What is the shortest route from Node 1 to Node 4?
1 to 2 to 4.

Table from previous page (repeated for convenience):

Iteration	1	2	3	4	5	6
1	0*	10	4**	15	17	7
2	0*	10	4*	13	16	5**
3	0*	10**	4*	13	16	5*
4	0*	10*	4*	12	11**	5*
5	0*	10*	4*	12**	11*	5*

(e) In the grid below fill in as many of the costs as you can using the information in the table above. (Put the value for $c(i,j)$ in the ith row and the j th column. If you do not have enough information, write "NEI" (for "not enough information"). Notice that I filled in the diagonal elements: $c(i,i) = 0$ for all i and below diagonal elements: $c(i,j) = \infty$ if $j > i$. In this table, I want direct costs (not the costs of a shortest route).

i/j	1	2	3	4	5	6
1	0	10	4	15	17	7
2	∞	0	NEI	2	1	NEI
3	∞	∞	0	9	12	1
4	∞	∞	∞	0	NEI	NEI
5	∞	∞	∞	∞	0	NEI
6	∞	∞	∞	∞	∞	0

(f) Is it possible to infer from the table the cost of the shortest route from Node 3 to Node 5? If so, what is the cost, what is the corresponding route, and why do you know it. If not, can you provide an upper bound for the cost? Why are you not sure whether it is actually the lowest cost?

There are two possible ways to get from 3 to 5. Either directly, which costs 12 or from 3 to 4 to 5, which costs 9 plus $c(4,5)$. So the minimum cost is between 9 and 12. I can't say more without knowing $c(4,5)$.

7. In each of the parts below, choose the single best answer.

   Parts (a)-(d) refer to the following information. Consider the following linear programming problems:

$$\max c \cdot x \text{ subject to } Ax \le b, x \ge 0 \qquad (\mathbf{P})$$
$$\max c' \cdot x \text{ subject to } A'x \le b', x \ge 0 \qquad (\mathbf{P'})$$

   where c and c' are (possibly different) vectors with n-components; b and b' are (possibly different) vectors with m-components; and A and A' are (possibly different) matrices with n columns and m rows. Let a_{ij} denote the entry of A in Row i and Column j and let a'_{ij} denote the entry of A' in Row i and Column j. Assume that these problems have unique solutions. Let x^* denote the solution to $\mathbf{P}$ and x'^* denote the solution to $\mathbf{P'}$.

   (a) If $b' = b$,

$$c'_j = \begin{cases} 2c_1, & \text{if } j = 1 \\ c_j, & \text{if } j \ne 1 \end{cases}$$

   and, for all i,

$$a'_{ij} = \begin{cases} 2a_{i1}, & \text{if } j = 1, \\ a_{ij}, & \text{if } j \ne 1, \end{cases}$$

   then

   i. $x'^* = x^*$.

   ii. $x'^*_j = \begin{cases} 2x_1, & \text{if } j = 1 \\ x_j, & \text{if } j \ne 1 \end{cases}$

   iii. $x'^*_j = \begin{cases} .5x_1, & \text{if } j = 1 \\ x_j, & \text{if } j \ne 1 \end{cases}$

   iv. None of the above (or insufficient information).

   (iii) is correct.

   (b) If $b' = b$, $A' = A$, and $c' = 2c$, then

   i. $x'^* = x^*$.

   ii. $x'^* = 2x^*$.

   iii. $x'^* = .5x^*$.

   iv. None of the above (or insufficient information)

   (i) is correct.

   (c) If $b' = b$, $A' = A$, and $c'_j = c_j + 1$ for all j, then

   i. $x'^* = x^*$.

   ii. $x'^*_j = x^*_j + 1$ for all j.

   iii. $x'^*_j = x^*_j - 1$ for all j.

iv. None of the above (or insufficient information).

(iv) is correct.

(d) If all of the entries in A and all of the components of b and c are integers, then

i. All of the components of x^* are integers.

ii. The value of **P** is an integer.

iii. Both (i) and (ii).

iv. None of the above (or insufficient information).

(iv) is correct. There are many counterexamples to the first two statements.

(e) Consider a network with weights c_{ij}, where the c_{ij} are nonnegative integers

i. The value of the associated minimum spanning tree problem is an integer.

ii. The value of the shortest route between any pair of nodes is an integer.

iii. The value of the maximum flow problem (for any designated source and sink) is an integer.

iv. Exactly two of the above.

v. All three of (i), (ii), and (iii).

vi. None of the above (or insufficient information).

(v) is correct. The algorithms give integer solutions when data are integers.