Econ 172A, Fall 2012: Final Examination (I)

Instructions.

1. The examination has seven questions. Answer them all.
2. If you do not know how to interpret a question, then ask me.
3. Questions 1-5 require you to provide detailed answers. You must explain how you arrive at your answer. If you properly use an algorithm introduced in the class that is appropriate for the problem, it is sufficient to say: "I used the algorithm introduced in class." If you use an alternate method, a detailed justification is necessary. Clear, accurate descriptions are necessary if you wish to receive partial credit for answers that contain computational errors.
4. To be clear: It is not sufficient to write down a correct numerical answer to receive credit. You must explain how you arrived at your answer and why it is appropriate.
5. No justification is needed for Questions 6 and 7.
6. The table below indicates how points will be allocated on the exam.
7. Work alone. You may not use notes, books, calculators, or any other electronic devices.
8. You have three hours.
9. Read the Buckley waiver (posted during exam). If you write the words "Buckley Waiver" at the top of this page <u>and</u> sign your name next to the w

	Score	Possible
I		30
II		20
III		20
IV		30
V		35
VI		40
VII		25
Exam Total		200
Course Total		400
Grade in Course		

1. The entries in the table below describe the costs associated with an assignment problem.

   There are four people (1, 2, 3, 4) and four jobs (A, B, C, and D). The entry in column j and row i is the cost associated with assigning person i to job j (so, for example, the cost of assigning person 3 to job C is 60). Find the cost-minimizing assignment of worker to job (each worker should do exactly one job; each job should be assigned to exactly one worker). You must justify your answer.

	A	B	C	D
1	40	9	10	4
2	100	85	102	98
3	80	70	60	50
4	45	10	12	30

2. Consider a marriage problem in which there are four men, (1, 2, 3, 4), and four woman (A, B, C, and D). The preferences of the men are:

   Suppose preferences are given by the following tables:

   Mens' Preferences

MEN				
1	C	D	B	A
2	A	D	B	C
3	B	A	C	D
4	B	D	A	C

   Womens' Preferences

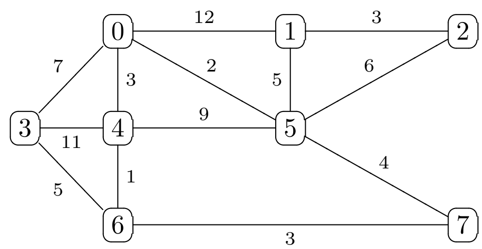
WOMAN				
A	1	2	3	4
B	1	4	3	2
C	2	4	3	1
D	4	1	2	3

   So, for example, Man 1 prefers Woman C to Woman D, Woman D to Woman B, and Woman B to Woman A.

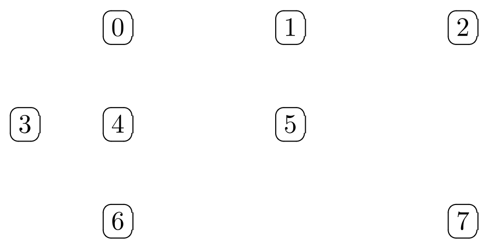
   (a) Exhibit a stable marriage for this problem. Justify your answer.

   (b) Is the stable marriage you found unique? If not, exhibit another stable marriage. If so, explain why it is unique.

3. Find a minimal spanning tree for the network below (the numbers on the edges are costs):



Draw a minimal spanning tree for the network above here (draw in the edges in the minimum spanning tree):



Briefly describe how you found this tree.

Compute the cost of the spanning tree that you found above:

4. Consider the network on the next page. (Note: I have included several copies of the network so that you can use it to show your work. Depending upon how you solve the problem, you may need fewer or more networks.) The numbers on the edges are capacities.

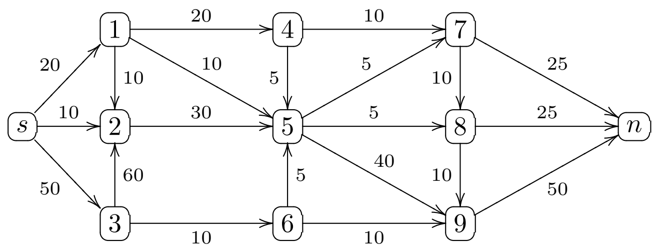
   (a) Find a maximum flow (from the source node (s) to the sink node (n)) for this network. You may describe the flow here or on one of the network diagrams on the next page. Please indicate your answer clearly. You must explain how you found the answer. If you properly used the algorithm introduced in class (and you show the steps), then you need no further justification. If you used another method, then you must explain the method and explain how it works.

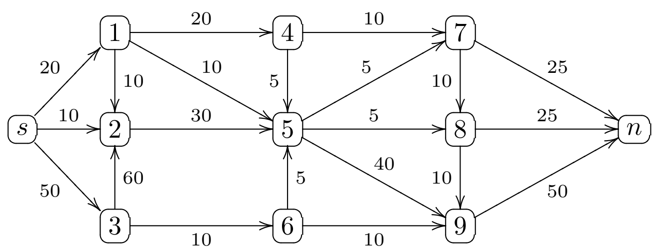
   (b) Find the value of the maximum flow.

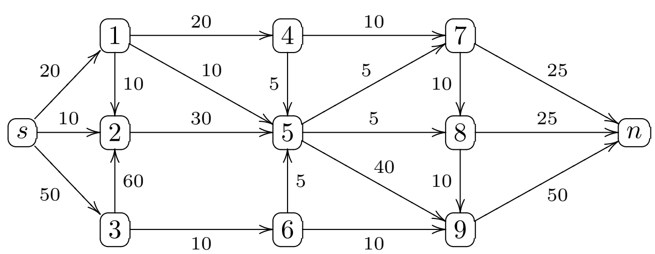
   (c) Find a minimum capacity cut for this network.

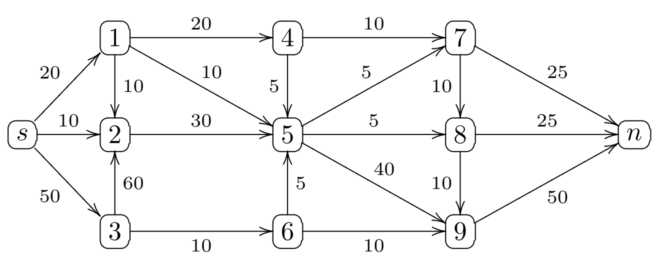
   (d) What is the capacity of the minimum capacity cut?

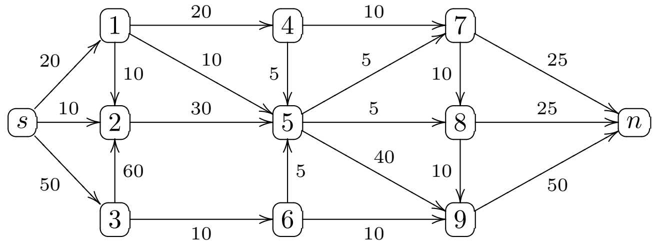
   (e) Find the capacity of the cut $\{s, 1, 2, 4, 5\}$ and $\{3, 6, 7, 8, 9, n\}$.

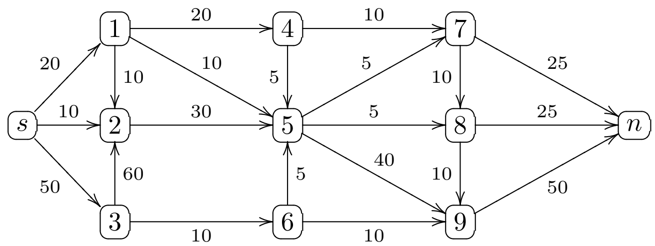

20
10
1
4
7
20
10
5
5
10
25
10
10
s
10
2
30
5
5
8
25
n
60
5
40
10
50
50
3
6
9
10
10

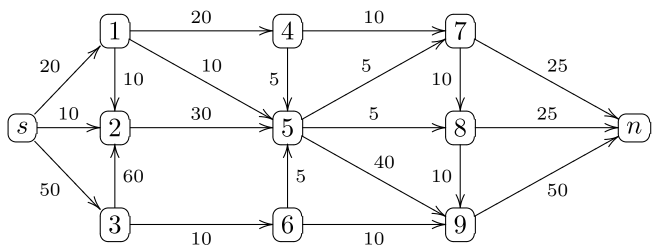

20
10
1
4
7
20
10
5
5
10
25
10
10
s
10
2
30
5
5
8
25
n
60
5
40
10
50
50
3
6
9
10
10

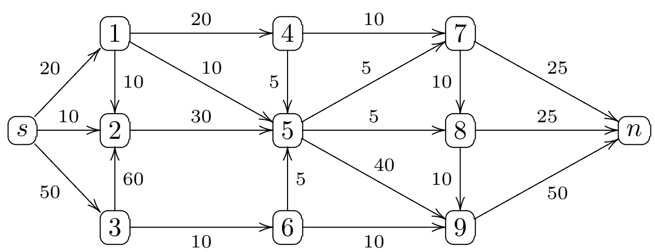

20
10
1
4
7
20
10
5
5
10
25
10
10
s
10
2
30
5
5
8
25
n
60
5
40
10
50
50
3
6
9
10
10


20
10
1
4
7
20
10
5
5
10
25
10
10
s
10
2
30
5
5
8
25
n
60
5
40
10
50
50
3
6
9
10
10


1
4
7
s
2
5
8
n
3
6
9
20
10
10
20
10
5
10
5
25
10
30
5
25
50
60
5
40
10
50
10
10


1
4
7
s
2
5
8
n
3
6
9
20
10
10
20
10
5
10
5
25
10
30
5
25
50
60
5
40
10
50
10
10


1
4
7
s
2
5
8
n
3
6
9
20
10
10
20
10
5
10
5
25
10
30
5
25
50
60
5
40
10
50
10
10


1
4
7
s
2
5
8
n
3
6
9
20
10
10
20
10
5
10
5
25
10
30
5
25
50
60
5
40
10
50
10
10

5. A drug company manufactures two kinds of sleeping pill: Pill 1 and Pill 2. The pills contain a mixture of two different chemicals, A and B (and no other ingredients). By weight, the Pill 1 must contain at least 65% Chemical A and Pill 2 must contain at least 55% Chemical A. There are two different ways to produce Chemicals A and B. Operating Process P for one hour requires 7 ounces of a raw material and 2 hours of labor time (because it must be supervised by 2 workers). Operating Process Q for one hour requires 5 ounces of a raw material and 3 hours of labor time. Operating Process P for one hour produces three ounces of each chemical. Operating Process Q for one hour produces 3 ounces of Chemical A and one ounce of Chemical B. There are $L > 0$ hours of labor and $R > 0$ ounces of raw material are available. The company earns a profit from Pill i of $\pi_i \geq 0$ dollars per ounce. The firm wishes to find a production plan that maximizes the profit it makes from manufacturing sleeping pills (subject to the constraints above).

   Formulate the problem as a linear programming problem.

   Your answer must include:

   (a) Descriptions of all of the relevant variables. Your description must include a specification of units.

   (b) A formula for the objective function.

   (c) Linear equations or inequalities representing every constraint.

6. The table below comes from a computation for a shortest-route problem, using the algorithm that I presented in class. (The table looks for shortest routes that begin at Node 1.) The nodes in the network are connected by directed edges that have non-negative costs. Assume that for all i, $c(i,i) = 0$. If $j > i$, there is no direct path from Node i to Node j. In these cases, $c(j,i) = \infty$.

Iteration/Node	1	2	3	4	5	6
1	0*	10	4**	15	17	7
2	0*	10	4*	13	16	5**
3	0*	10**	4*	13	16	5*
4	0*	10*	4*	12	11**	5*
5	0*	10*	4*	12**	11*	5*

(a) What is the cost of the shortest route from Node 1 to Node 5?

(b) What is the shortest route from Node 1 to Node 5?

(c) What is the cost of the shortest route from Node 1 to Node 4?

(d) What is the shortest route from Node 1 to Node 4?

Table from previous page (repeated for convenience):

Iteration	1	2	3	4	5	6
1	0*	10	4**	15	17	7
2	0*	10	4*	13	16	5**
3	0*	10**	4*	13	16	5*
4	0*	10*	4*	12	11**	5*
5	0*	10*	4*	12**	11*	5*

(e) In the grid below fill in as many of the costs as you can using the information in the table above. (Put the value for $c(i,j)$ in the ith row and the j th column. If you do not have enough information, write "NEI" (for "not enough information"). Notice that I filled in the diagonal elements: $c(i,i) = 0$ for all i and below diagonal elements: $c(i,j) = \infty$ if $j > i$. In this table, I want direct costs (not the costs of a shortest route).

i/j	1	2	3	4	5	6
1	0					
2	∞	0				
3	∞	∞	0			
4	∞	∞	∞	0		
5	∞	∞	∞	∞	0	
6	∞	∞	∞	∞	∞	0

(f) Is it possible to infer from the table the cost of the shortest route from Node 3 to Node 5? If so, what is the cost, what is the corresponding route, and why do you know it. If not, can you provide an upper bound for the cost? Why are you not sure whether it is actually the lowest cost?

7. In each of the parts below, choose the single best answer.

Parts (a)-(d) refer to the following information. Consider the following linear programming problems:

$$\max c \cdot x \text{ subject to } Ax \leq b, x \geq 0 \qquad (\mathbf{P})$$
$$\max c' \cdot x \text{ subject to } A'x \leq b', x \geq 0 \qquad (\mathbf{P'})$$

where c and c' are (possibly different) vectors with n-components; b and b' are (possibly different) vectors with m-components; and A and A' are (possibly different) matrices with n columns and m rows. Let a_{ij} denote the entry of A in Row i and Column j and let a'_{ij} denote the entry of A' in Row i and Column j. Assume that these problems have unique solutions. Let x^* denote the solution to **P** and x'^* denote the solution to **P**$'$.

(a) If $b' = b$,

$$c'_j = \begin{cases} 2c_1, & \text{if } j = 1 \\ c_j, & \text{if } j \neq 1 \end{cases}$$

and, for all i,

$$a'_{ij} = \begin{cases} 2a_{i1}, & \text{if } j = 1, \\ a_{ij}, & \text{if } j \neq 1, \end{cases}$$

then

i. $x'^* = x^*$.

ii. $x'^*_j = \begin{cases} 2x_1, & \text{if } j = 1 \\ x_j, & \text{if } j \neq 1 \end{cases}$

iii. $x'^*_j = \begin{cases} .5x_1, & \text{if } j = 1 \\ x_j, & \text{if } j \neq 1 \end{cases}$

iv. None of the above (or insufficient information).

I doubled all of the coefficients associated with x_j. This is equivalent to changing units. (If x_j was measured in pounds in the original problem, it is measured in 2 pounds in the primed problem.) Hence all you need to do is divide the x_1 value in half to get the new solution. Correct answer: c.

(b) If $b' = b$, $A' = A$, and $c' = 2c$, then

i. $x'^* = x^*$.

ii. $x'^* = 2x^*$.

iii. $x'^* = .5x^*$.

iv. None of the above (or insufficient information)

Doubling the resources permits you to double the inputs. (b) is correct.

(c) If $b' = b$, $A' = A$, and $c'_j = c_j + 1$ for all j, then

i. $x'^* = x^*$.

ii. $x'^*_j = x^*_j + 1$ for all j.

iii. $x'^*_j = x^*_j - 1$ for all j.

iv. None of the above (or insufficient information).

(d) is correct.

(d) If all of the entries in A and all of the components of b and c are integers, then

i. All of the components of x^* are integers.

ii. The value of **P** is an integer.

iii. Both (i) and (ii).

iv. None of the above (or insufficient information).

(d) is correct. (There are many examples where the first two things don't happen.)

(e) Consider a network with weights c_{ij}, where the c_{ij} are nonnegative integers

i. The value of the associated minimum spanning tree problem is an integer.

ii. The value of the shortest route between any pair of nodes is an integer.

iii. The value of the maximum flow problem (for any designated source and sink) is an integer.

iv. Exactly two of the above.

v. All three of (i), (ii), and (iii).

vi. None of the above (or insufficient information).

(e) is correct. (The algorithms demonstrate this.)