

# Econ 172A - Slides from Lecture 8

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## Announcements

- ▶ Important: Midterm seating assignments. Posted tonight.
- ▶ Corrected Answers to Quiz 1 posted.
- ▶ Quiz 2 on Thursday at end of class. (What to expect: True and False Questions on Duality)
- ▶ Midterm on November 1, 2012.
- ▶ Problems for Quiz 2: Focus on 2010 Quiz 2 (and related duality questions).
- ▶ Problems for Midterm:
  1. 2004: Midterm #1: 2, 3; Midterm #2. Problem Set 1, Problem Set 2 - 1,2. Final: 2, 3, 5.
  2. 2007: Midterm. Problem Sets 1 and 2.
  3. 2008: Midterm. Problem Sets 1 and 2. Final: 5, 7 a, b.
  4. 2010: Midterm. Quizzes 1, 2. Final 1 a,b; 4

## Changing Right-Hand Side of Non-Binding Constraint

- ▶ Dual prices capture the effect of a change in the amount of resources.
- ▶ **Observation** Increasing the amount of resource in a non-binding constraint, does not change the solution.
- ▶ Small decreases do not change anything.
- ▶ If you decreased the amount of resource enough to make the constraint binding, your solution could change.
- ▶ Similar to changing the coefficient of a non-basic variable in the objective function.

## Changing Right-Hand Side of Binding Constraint

- ▶ Changes in the right-hand side of binding constraints always change the solution (the value of  $x$  must adjust to the new constraints).
- ▶ Dual variable associated with the constraint measures how much the objective function will be influenced by the change.

## Adding a Constraint

1. If you add a constraint to a problem, two things can happen.
2. Your original solution satisfies the constraint or it doesn't.
3. If it does, then you are finished. If you had a solution before and the solution is still feasible for the new problem, then you must still have a solution.
4. If the original solution does not satisfy the new constraint, then possibly the new problem is infeasible.
5. If not, then there is another solution.
6. The value must go down.
7. If your original solution satisfies your new constraint, then you can do as well as before.
8. If not, then you will do worse.

## Relationship to the Dual

- ▶ The objective function coefficients correspond to the right-hand side constants of resource constraints in the dual.
- ▶ The primal's right-hand side constants correspond to objective function coefficients in the dual.
- ▶ Hence the exercise of changing the objective function's coefficients is really the same as changing the resource constraints in the dual.

## Understanding Sensitivity Information Provided by Excel

1. Excel permits you to create a sensitivity report with any solved LP.
2. The report contains two tables, one associated with the variables and the other associated with the constraints.

## Sensitivity Information on Changing (or Adjustable) Cells

The top table in the sensitivity report refers to the variables in the problem.

1. The first column (Cell) tells you the location of the variable in your spreadsheet.
2. The second column tells you its name (if you named the variable).
3. The third column tells you the final value.
4. The fourth column is called the reduced cost.
5. The fifth column tells you the coefficient in the problem.
6. The final two columns are labeled “allowable increase” and “allowable decrease.”

## Allowable Increase and Decrease

These comments refer to standard example. The information appears on the spreadsheet “Examples” listed in Section VII of the notes on the web page.

The allowable increase is the amount by which you can increase the coefficient of the objective function without causing the optimal basis to change. The allowable decrease is the amount by which you can decrease the coefficient of the objective function without causing the optimal basis to change.

- ▶ First row of the table describes the variable  $x_1$ .
- ▶ The coefficient of  $x_1$  in the objective function is 2.
- ▶ The allowable increase is 9, the allowable decrease is “1.00E+30.” which means  $10^{30}$ , which **really** means  $\infty$ .
- ▶ This means that provided that the coefficient of  $x_1$  in the objective function is less than  $11 = 2 + 9 = \text{original value} + \text{allowable increase}$ , the basis (variables that are positive in the solution) does not change.
- ▶ Since  $x_1$  is a non-basic variable, when the basis stays the

## Explanations -Non Basic

- ▶ If you lower the objective coefficient of a non-basic variable, then your solution does not change.
- ▶ The allowable decrease will always be infinite for a non-basic variable.
- ▶ Increasing the coefficient of a non-basic variable may lead to a change in basis.
- ▶ If you increase the coefficient of  $x_1$  from 2 to anything greater than 9 (that is, if you add more than the allowable increase of 7 to the coefficient), then you change the solution.
- ▶ The sensitivity table does not tell you how the solution changes (common sense suggests that  $x_1$  will take on a positive value).

- ▶ The line associated with the other non-basic variable of the example,  $x_3$ , is remarkably similar.
- ▶ The objective function coefficient is different (3 rather than 2), but the allowable increase and decrease are the same as in the row for  $x_1$ .
- ▶ It is a coincidence that the allowable increases are the same.
- ▶ It is no coincidence that the allowable decrease is the same.
- ▶ The solution of the problem does not change as long as the coefficient of  $x_3$  in the objective function is less than or equal to 10.

## Basic Variables

- ▶ For  $x_2$  the allowable increase is infinite while the allowable decrease is 2.69 (it is  $2\frac{9}{13}$  to be exact).
- ▶ The solution won't change if you increase the coefficient of  $x_2$ , but it will change if you decrease the coefficient enough (that is, by more than 2.7).
- ▶ The solution does not change no matter how much you increase  $x_2$ 's coefficient because there is no way to make  $x_2 > 10.4$  and still satisfy the constraints of the problem.
- ▶ The solution does change when you increase  $x_2$ 's coefficient by enough means that there is a feasible basis in which  $x_2$  takes on a value lower than 10.4.
- ▶ The range for  $x_4$  is different.
- ▶ Line four of the sensitivity table says that the solution of the problem does not change provided that the coefficient of  $x_4$  in the objective function stays between 16 (allowable increase 15 plus objective function coefficient 1) and -4 (objective function coefficient minus allowable decrease).

- ▶ If you make  $x_4$  sufficiently more attractive, then your solution will change to permit you to use more  $x_4$ .
- ▶ If you make  $x_4$  sufficiently less attractive the solution will also change. (Now you'd use less  $x_4$ . )
- ▶ When you change the coefficient of a basic variable, the value of the problem will change.
- ▶ You can use the table to tell you the solution of the LP when you take the original constraints and replace the original objective function by

$$\max 2x_1 + 6x_2 + 3x_3 + x_4$$

(that is, you change the coefficient of  $x_2$  from 4 to 6), then the solution to the problem remains the same.

- ▶ The value of the solution changes because now you multiply the 10.4 units of  $x_2$  by 6 instead of 4.
- ▶ The objective function therefore goes up by 20.8.

## Reduced Cost

- ▶ Reduced cost of a variable: smallest change in the objective function coefficient needed to arrive at a solution in which the variable takes on a positive value when you solve the problem.
- ▶ Meaning??
- ▶ Algebraic Definition (Non-basic): The reduced cost of a non-basic variable is the negative of the allowable increase (that is, if you change the coefficient of  $x_1$  by  $-7$ , then you arrive at a problem in which  $x_1$  takes on a positive value in the solution).
- ▶ Economic Interpretation: The allowable increase in the coefficient is 7.
- ▶ The reduced cost of a basic variable is always zero.
- ▶ Economic Interpretation: You need not change the objective function at all to make the variable positive.

## More Reduced Cost

- ▶ You can figure out reduced costs from the other information in the table: If the final value is positive, then the reduced cost is zero.
- ▶ Warning (for experts, not exam material): The previous statement fails in “degenerate” cases. For details, see me.
- ▶ If the final value of a variable is zero, then the reduced cost is negative one times the allowable increase.
- ▶ The reduced cost of a variable is also the amount of slack in the dual constraint associated with the variable.
- ▶ With this interpretation, complementary slackness implies that if a variable that takes on a positive value in the solution, then its reduced cost is zero.

## Sensitivity Information on Constraints

- ▶ Second sensitivity table.
- ▶ The cell column identifies the location of the left-hand side of a constraint.
- ▶ The name column gives its name (if any).
- ▶ The final value is the value of the left-hand side when you plug in the final values for the variables.
- ▶ The shadow price is the dual variable associated with the constraint.
- ▶ The constraint R.H. side is the right hand side of the constraint.
- ▶ The allowable increase tells you by how much you can increase the right-hand side of the constraint without changing the basis.
- ▶ The allowable decrease tells you by how much you can decrease the right-hand side of the constraint without changing the basis.

## Complementary Slackness Again

- ▶ Complementary Slackness guarantees a relationship between the columns in the constraint table.
- ▶ The difference between the “Constraint Right-Hand Side” column and the “Final Value” column is the slack.
- ▶ For example, the slack for the three constraints is 0 ( $= 12 - 12$ ), 37 ( $= 7 - (-30)$ ), and 0 ( $= 10 - 10$ ), respectively.
- ▶ Complementary Slackness says that if there is slack in the constraint then the associated dual variable is zero.
- ▶ Hence CS tells us that the second dual variable must be zero.

- ▶ You can figure out information on allowable changes from other information in the table.
- ▶ The allowable increase and decrease of non-binding variables can be computed knowing final value and right-hand side constant.
- ▶ If a constraint is not binding, then adding more of the resource is not going to change your solution.
- ▶ Hence the allowable increase of a resource is infinite for a non-binding constraint.
- ▶ Similarly: the allowable increase of a resource is infinite for a constraint with slack.
- ▶ So in example allowable increase of the second constraint is infinite.

- ▶ Also: The allowable decrease of a non-binding constraint is equal to the slack in the constraint.
- ▶ Hence the allowable decrease in the second constraint is 37.
- ▶ If you decrease the right-hand side of the second constraint from its original value (7) to anything greater than  $-30$  you do not change the optimal basis.
- ▶ The only part of the solution that changes is that the value of the slack variable for this constraint.
- ▶ If you solve an LP and find that a constraint is not binding, then you can remove all of the unused (slack) portion of the resource associated with this constraint and not change the solution to the problem.

## Allowable Increase and Decrease

- ▶ First constraint. If the right-hand side of the first constraint is between 10 (original value 12 minus allowable decrease 2) and infinity, then the basis of the problem does not change.
- ▶ Solution usually does change.
- ▶ Saying that the basis does not change means that the variables that were zero in the original solution continue to be zero in the new problem (with the right-hand side of the constraint changed).
- ▶ When the amount of available resource changes, necessarily the values of the other variables change.
- ▶ In diet problem, getting requiring more nutrient of something that you had supplied exactly before leads you to buy more food.
- ▶ In production problem, getting more of a scarce ingredient allows you to produce more.
- ▶ Changes within the allowable range for a binding constraint's RHS doesn't change the positive variables in the solution, but

## Third Constraint

- ▶ The values for the allowable increase and allowable decrease say that the basis that is optimal for the original problem (when the right-hand side of the third constraint is equal to 10) remains obtain provided that the right-hand side constant in this constraint is between  $-2.3333$  and  $12$ .
- ▶ Suppose that your LP involves four production processes and uses three basic ingredients.
- ▶ Call the ingredients land, labor, and capital.
- ▶ The outputs vary use different combinations of the ingredients.
- ▶ Maybe they are growing fruit (using lots of land and labor), cleaning bathrooms (using lots of labor), making cars (using lots of labor and and a bit of capital), and making computers (using lots of capital).
- ▶ For the initial specification of available resources, you find that your want to grow fruit and make cars.

- ▶ If you get an increase in the amount of capital, you may wish to shift into building computers instead of cars.
- ▶ If you experience a decrease in the amount of capital, you may wish to shift away from building cars and into cleaning bathrooms instead.

## Duality

- ▶ The “Adjustable Cells” table and the “Constraints” table provide the same information.
- ▶ Dual variables correspond to primal constraints.
- ▶ Primal variables correspond to dual constraints.
- ▶ The “Adjustable Cells” table tells you how sensitive primal variables and dual constraints are to changes in the primal objective function.
- ▶ The “Constraints” table tells you how sensitive dual variables and primal constraints are to changes in the dual objective function (right-hand side constants in the primal).