Econ 172A - Slides from Lecture 7

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Announcements

- Be prepared for midterm room/seating assignments.
- Quiz 2 on October 25, 2012. (Duality, up to, but not including sensitivity analysis.)
- Midterm on November 1, 2012.
- Problems: 10-5: Problems on current material: Problem Set 2, 2004: #1, #4; 2007: #1; 2008: #2.
  Midterm 1, 2004: all; Midterm 2007: #1, 4; Midterm 1, 2008: #2
  On upcoming material:
  Problem Set 2: 2004: #2, #3; 2007: #2, #3; 2008: #1, #3
  Midterm 2 2004: all; 2007: 5; 2008 (midterm 2) all.
  Final 2004: 3, 5; 2008: 5,6

Duality Theorem Says

1. The value of the original problem must be no greater than the value of the dual.
2. In fact, the values are equal.
3. Dual Variables are prices.
4. The prices may have nothing to do with market supply and demand.
5. They are prices in the sense that they value the inputs to the production process.
6. That is, they provide answers to questions like: How much is the first ingredient worth to you (if you have access to the technology that turns inputs into outputs according to the matrix A, b is your list of available inputs, and c are the prices at which you can sell final outputs). If A, b, or c changes, then you would expect the "price" of ingredients to change too.

Repeat

- If someone offers to buy a little bit of the first input, then you would accept if you are offered y_1 or more (per unit).
- If you could buy a more of the first input at less than y_1, it would be profitable for you to do so.
- The identity between the value of the primal and the value of the dual tells you that the ingredients, when evaluated according to the prices y, are worth exactly as much as the final product.

Complementary Slackness

- If you don't use all of the supply of the first resource (slack in Primal constraint 1), you would gladly a bit of that resource at *any* positive price.
- CS says that the resource is worth nothing to you. $y_1 = 0$.
- Suppose that you produce positive amounts of the *n*th good when you solve the problem ($x_n > 0$).
- The dual constraint says that the amount you can sell the *n*th good for is no greater than the value of its ingredients.
- If you really could sell the ingredients, then you would do so (instead of setting $x_n > 0$) – unless the *n*th constraint of the dual is binding.

There is More

- Suppose $y_1 > 0$.
- You are willing to pay for more of ingredient 1.
- Hence, you use up all the ingredient one available.
- First primal constraint binds.
- Suppose the jth dual constraint is not binding.
- You can sell inputs to the jth production process for more than you can sell the output.
- You don't produce any output.

Interpretation of the Dual

Primal as Production Problem.
Dual as Buyout Problem.

Dual Variables as Prices

- Mathematicians call them dual variables.
- More generally (in 172B) they will be called Lagrange multipliers (or just multipliers).
- Economists call them dual prices, shadow prices, or implicit prices.

Why Prices?

- Diet problem: ith dual variable was the price of the ith nutrient.
- How is it a price? y_i represented how much the pill seller would charge for one unit of the ith nutrient.
- Not a market price.
- Prices are implicit because they describe imaginary transactions.
- Dual prices depend on "context."
- The cost of nutrients will depend on the price of the different foods.
- The value of nutrients would change if the price of a nutrient rich food rose (this would tend to raise nutrient prices).
- The discovery of a new, cheap nutrient rich food might lower nutrient prices.
- Changes in nutritional requirements might change dual prices.

Production Problem

- Dual variable price of ingredient.
- Price describes how much producer would pay for an additional unit.
- Hence price is zero if not using all available ingredient.
- Not (necessarily) market price.
- Price will depend on technology.
- Prices will help evaluate alternative production possibilities.

Some Supporting Algebra

- Start with LP is standard form. (max, $\leq$ constraints, non neg variables).
- Solve it and its dual.
- Call this the old problem and denote the solutions x^{old} and y^{old} and the associated value V^{old}.
- It follows that

$$V^{old} = c^{old} \cdot x^{old} = b^{old} \cdot y^{old}.$$

New Problem

- Create a new problem and find the solution to it.
- Call this the new problem and denote the solutions x^{new} and y^{new} and the associated value V^{new}.
- Again you have

$$V^{new} = c^{new} \cdot x^{new} = b^{new} \cdot y^{new}.$$

Comparisons

- Assume: you get from the old problem to the new problem by changing one of the resource constants, b_i, by adding Δ to it (Δ could be positive or negative).
- Assume that when you make this change, the solution to the dual does not change, that is $y^{new} = y^{old}$.
- The first assumption is a description of how you changed the problem (harmless).
- The second is "usually" satisfied: when you change the resource constants in the primal, you usually do not change the solution to the dual (but you do typically change the *value* of the solution).
- Why should we believe the previous claim?
- But: resource constraints typically do change the solution to the primal. For example, if you learned that the government requires that you eat less Vitamin C, you may buy fewer oranges.

PUNCHLINE

- The change in the value of the problems, $V^{new} - V^{old}$, is equal to
$$b^{new} \cdot y^{new} - b^{old} \cdot y^{old}.$$
- If the dual variables do not change we can write $y^{new} = y^{old} = y$. Therefore,
$$V^{new} - V^{old} = (b^{new} - b^{old}) \cdot y = \Delta y_i.$$
- The equation comes from the assumption that you get b^{new} from b^{old} by adding Δ to the ith component of b^{old}, while leaving the other components unchanged.
- This the algebraic expression of my interpretation of dual variables.
- The difference in value as you go from old to new problem is equal to the change in the resource constant times the associated dual variable.
- You can figure out the change in value without solving the new LP.

Techicalities

- The important assumption is that when you change the LP, you do not change the solution to the dual.
- One expects that "small" changes in b_i do not change the solution to the dual.
- The restriction that the dual solution does not change means that the interpretation of dual variables remains valid only for "small" changes in the level of resources available.
- When computer programs solve LPs, you get precise information on the meaning of "small."

Intuition on why small changes in RHS don't change dual solution

- RHS in primal and objective function in Primal.
- What happens when you change objective function?
- Think graphically.
- If solution is unique, then changing slope of objective function won't change location of solution.

Interpreting the Dual: Watch your Units

- Since the values of primal and dual are equal, they have the same units.
- Typically, values come in units of money.
- Diet problem: the value of the primal is the amount you spend on food. The value of the dual is the amount the pill seller can earn selling pills.
- Production / buyout problem: Value of primal is revenue. Value of dual is cost of ingredients.
- In both cases, the units are monetary units.
- Since you know units on basic data of the problem (A, b, and c), you can figure out units of dual variables.
- This does not provide a description of the dual, but it does give you a start. It also allows you to recognize nonsense (inappropriate units) when you see them.

Sensitivity Analysis

- All models involve approximations.
- In particular, A, b, or c may be approximations.
- Or information may change.
- **Sensitivity analysis** is a systematic study of how solutions respond to (small) changes in the data.
- Goal, answer questions like:
  1. If the objective function changes, how does the solution change?
  2. If resources available change, how does the solution change?
  3. If a constraint is added to the problem, how does the solution change?

One Approach and Our Approach

Solve LPs over and over again.

If you think that the price of your primary output will be between \$100 and \$120 per unit, you can solve one for each whole number between \$100 and \$120.

Take full advantage of the structure of LP programming problems and their solution.

It turns out that you can often figure out what happens in "nearby" LP problems just by thinking and by examining the information provided by the simplex algorithm.

Same old example ...

- In sensitivity analysis you evaluate what happens when only one parameter of the problem changes. That is, one output price goes up or one resource quantity goes down.
- Familiar example:

$$
\begin{array}{lrcrcrcrcr}
\max & 2x_1 & + & 4x_2 & + & 3x_3 & + & x_4 & & \\
\text{subject to} & 3x_1 & + & x_2 & + & x_3 & + & 4x_4 & \leq & 12 \\
 & x_1 & - & 3x_2 & + & 2x_3 & + & 3x_4 & \leq & 7 \\
 & 2x_1 & + & x_2 & + & 3x_3 & - & x_4 & \leq & 10 \\
 & & & & & & & x & \geq & 0
\end{array}
$$

We know that the solution to this problem is $x_0 = 42, x_1 = 0; x_2 = 10.4; x_3 = 0; x_4 = .4.$

Changing Objective Function

- You solve LP. Solution is x^*.
- Someone gives you new LP.
- New LP same as old except: objective function coefficient changes.
- Two cases.
- Case 1: You change the coefficient associated with x_j, where $x_j^* = 0$. (x_j is called a non-basic variable.)
  In the example, the relevant non-basic variables are x_1 and x_3.
- Case 2: You change the coefficient associated with x_j, where $x_j^* \neq 0$. (x_j is called a basic variable.)
  In the example, the relevant basic variables are x_2 and x_4.

Non-Basic Variable Coefficient Goes Down

- For example, suppose that the coefficient of x_1 in the objective function above was reduced from 2 to 1 (so that the objective function is: $\max x_1 + 4x_2 + 3x_3 + x_4$).
- You have taken a variable that you didn't want to use in the first place (you set $x_1 = 0$) and then made it less profitable (lowered its coefficient in the objective function).
- You are still not going to use it.
- The solution does not change.
- The value does not change.

Observation If you lower the objective function coefficient of a non-basic variable, then the solution does not change.

Non-Basic Variable Coefficient Goes Up

- Intuitively, raising it just a little bit should not matter, but raising the coefficient a lot might induce you to change the value of x in a way that makes $x_1 > 0$.
- Expect a solution to continue to be valid for a range of values for coefficients of non-basic variables.
- The range should include all lower values for the coefficient and some higher values.
- If the coefficient increases enough (and putting the variable into the basis is feasible), then the solution changes.

Basic Variable Coefficient Goes Down

- What happens to your solution if the coefficient of a basic variable (like x_2 or x_4 in the example) decreases?
- Different from before.
- The change makes the variable contribute less to profit.
- Hence your value must decrease.
- You should expect that a sufficiently large reduction makes you want to change your solution (and lower the value the associated variable).
- If the coefficient of x_2 in the objective function in the example were 2 instead of 4 (so that the objective was $\max 2x_1 + 2x_2 + 3x_3 + x_4$), maybe you would want to set $x_2 = 0$ instead of $x_2 = 10.4$.
- A small reduction in x_2's objective function coefficient would not cause you to change your solution.
- Any change will change the **value** of your objective function.

Continued

You compute the value by plugging in x into the objective function, if $x_2 = 10.4$ and the coefficient of x_2 goes down from 4 to 2, then the contribution of the x_2 term to the value goes down from 41.6 to 20.8 (assuming that the solution remains the same).

Basic Variable Coefficient Goes Up

- Value goes up.
- You still want to use the variable ($x_j > 0$ after change).
- If coefficient goes up enough, you might want to use even more of x_j.
- Intuitively, there should be a range of values of the coefficient of the objective function (a range that includes the original value) in which the solution of the problem does not change.
- Outside of this range, the solution will change (to lower the value of the basic variable for reductions and increase its value of increases in its objective function coefficient).
- The value of the problem always changes when you change the coefficient of a basic variable.

Changing Right-Hand Side of Non-Binding Constant

- Dual prices capture the effect of a change in the amount of resources.
- **Observation** Increasing the amount of resource in a non-binding constraint, does not change the solution.
- Small decreases do not change anything.
- If you decreased the amount of resource enough to make the constraint binding, your solution could change.
- Similar to changing the coefficient of a non-basic variable in the objective function.

Changing Right-Hand Side of Binding Constraint

- Changes in the right-hand side of binding constraints always change the solution (the value of x must adjust to the new constraints).
- Dual variable associated with the constraint measures how much the objective function will be influenced by the change.

Adding a Constraint

1. If you add a constraint to a problem, two things can happen.
2. Your original solution satisfies the constraint or it doesn't.
3. If it does, then you are finished. If you had a solution before and the solution is still feasible for the new problem, then you must still have a solution.
4. If the original solution does not satisfy the new constraint, then possibly the new problem is infeasible.
5. If not, then there is another solution.
6. The value must go down.
7. If your original solution satisfies your new constraint, then you can do as well as before.
8. If not, then you will do worse.

Relationship to the Dual

- The objective function coefficients correspond to the right-hand side constants of resource constraints in the dual.
- The primal's right-hand side constants correspond to objective function coefficients in the dual.
- Hence the exercise of changing the objective function's coefficients is really the same as changing the resource constraints in the dual.