Econ 172A - Slides from Lecture 6

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October 16, 2012

Announcements

- Quiz Answers posted after class. Graded Quizzes Available in Section.
- 40 points possible, median 27.
- There is no such thing as a trick question, but quizzes do not give graders an opportunity to see what you are thinking.
- Quiz 2 on October 25, 2012.
- Midterm on November 1, 2012.
- There will be assigned seats for the midterm.
- Problems: 10-5: Problems on current material: Problem Set 2, 2004: #1, #4; 2007: #1; 2008: #2.
  Midterm 1, 2004: all; Midterm 2007: #1, 4; Midterm 1, 2008: #2

Theorem (Complementary Slackness)

Assume problem (P) has a solution x^* and problem (D) has a solution y^*.

1. *If $x_j^* > 0$, then the jth constraint in (D) is binding.*
2. *If the jth constraint in (D) is not binding, then $x_j^* = 0$.*
3. *If $y_i^* > 0$, then the ith constraint in (P) is binding.*
4. *If the ith constraint in (P) is not binding, then $y_i^* = 0$.*

What if guess is not a solution?

Take the LP:

$$\begin{array}{lrcrcl} \max & x_1 & - & x_2 & & \\ \text{subject to} & -2x_1 & + & x_2 & \leq & 2 \\ & x_1 & - & 2x_2 & \leq & 2 \\ & x_1 & + & x_2 & \leq & 5 \\ & & & x & \geq & 0 \end{array}$$

.

The dual is:

$$\begin{array}{lrcrcrcr} \min & 2y_1 & + & 2y_2 & + & 5y_3 & & \\ \text{subject to} & -2y_1 & + & y_2 & + & y_3 & \geq & 1 \\ & y_1 & - & 2y_2 & + & y_3 & \geq & -1 \\ & & & & & y & \geq & 0 \end{array}$$

Does $(1, 4)$ solves Primal?

- You could solve by graphing.
- You can solve by computer.
- You can use CS.
- Check feasibility of $(1, 4)$ in the primal.
- If it is not feasible, then it can't be a solution.

- $(1, 4)$ is feasible.
- First constraint binds. This tells you nothing about dual variables.
- Second constraint holds with slack. Hence $y_2 = 0$ must hold in solution to dual.
- $(1, 4)$ satisfies both non-negativity constraints strictly.
- Hence both dual constraints bind.
- So dual "reduces" to:

$$-2y_1 + y_3 = 1$$

$$y_1 + y_3 = -1$$

$$y_2 = 0$$

- Conclude: $y_1 = -\frac{2}{3}$ and $y_3 = -\frac{1}{3}$.

Punchline

- Assuming that $(1, 4)$ solves the primal leads to the conclusion that $y = (-\frac{2}{3}, 0, -\frac{1}{3})$ solves the dual.
- But: this choice of y is not feasible for the dual (it violates the non-negative constraints).
- Hence $(1, 4)$ did not solve the primal after all.
- Using the same kind of reasoning, you can check that the solution to the primal is $(4, 1)$. The dual solution is $y = (0, \frac{2}{3}, \frac{1}{3})$. The value of both problems is 3.

Generally

1. Start with a "guess" for the primal.
2. Check feasibility.
3. If infeasible, then it cannot be a solution.
4. If feasible, use CS conditions to reduce dual to a system of equations.
5. Solve these equations to get guesses for dual variables.
6. Check feasibility of the dual guess.
7. If the dual guess is feasible, then both the original primal guess and the dual guess actually solve their problems.
8. If the dual guess is not feasible, then neither guess is a solution.
9. In the example, we concluded that y was not feasible because it violated the non-negativity constraints.
10. It is possible that the guess you generate for the dual is infeasible because it violates one of the resource constraints in the dual.

Using Complementary Slackness to Solve Duals, Continued

Let's take a familiar example:

$$\begin{array}{rrcrcrcrcr}
\max & 2x_1 & + & 4x_2 & + & 3x_3 & + & x_4 & & \\
\text{subject to} & 3x_1 & + & x_2 & + & x_3 & + & 4x_4 & \leq & 12 \\
& x_1 & - & 3x_2 & + & 2x_3 & + & 3x_4 & \leq & 7 \\
& 2x_1 & + & x_2 & + & 3x_3 & - & x_4 & \leq & 10 \\
& & & & & & & x & \geq & 0
\end{array}$$

We confirmed that the solution to this problem is $x_0 = 42, x_1 = 0; x_2 = 10.4; x_3 = 0; x_4 = .4$. We can use this information (and Complementary Slackness conditions) to solve the dual. First write the dual:

$$\begin{array}{rrcrcrcr}
\min & 12y_1 & + & 7y_2 & + & 10y_3 & & \\
\text{subject to} & 3y_1 & + & y_2 & + & 2y_3 & \geq & 2 \\
& y_1 & - & 3y_2 & + & y_3 & \geq & 4 \\
& y_1 & + & 2y_2 & + & 3y_3 & \geq & 3 \\
& 4y_1 & + & 3y_2 & - & y_3 & \geq & 1 \\
& & & & & y & \geq & 0
\end{array}$$

ONE MORE

Does $y_1 = .2, y_2 = 1.4, y_3 = 0$ solve dual of big problem?

1. Is it feasible?
2. First dual constraint binds. Says nothing about what primal solution looks like.
3. Second dual constraint fails. Stop.
4. Conclude: $(.2, 1.4, 0)$ doesn't solve dual.

HOW ABOUT

Does $x_1 = 1.6, x_2 = x_3 = 0, x_4 = 1.8$ solve the big problem?

1. Is it feasible?
2. First constraint binds. Says nothing about what dual solution looks like.
3. Second constraint binds. Says nothing about what dual solution looks like.
4. Third constraint slack. Says that $y_3 = 0$.
5. Since $x = (1.6, 0, 0, 1.8)$ satisfies nonnegativity constraint, it is feasible.
6. Conclude that if x solves Primal that solution to dual involves $y_3 = 0$ and first and fourth dual constraints binding (because $x_1, x_4 > 0$).

Hence dual must satisfy:

$$3y_1 + y_2 = 2$$

$$4y_1 + 3y_2 = 1$$

$$y_3 = 0$$

So $(y_1, y_2, y_3) = (1, -1, 0)$.
This violates non-negativity. Hence original primal guess was not a solution.

Warning

If it turned out that the solution to the equations was non-negative, we would still have had to check to make sure that the omitted constraints ($y_1 - 3y_2 + y_3 \geq 4$ and $y_1 + 2y_2 + 3y_3 \geq 3$) hold. If they did, then the primal guess would be a solution.

An Example: Production Problem

- n different production activities.
- If you operate activity j at unit level, then you can sell it for c_j.
- m different basic resources.
- You have the amount b_i of resource i.
- When you operate activity j at unit level, you use up some of the basic resources.
- The technology matrix A describes this information.
- a_{ij} of the matrix A is the amount of basic resource i needed to operate the jth activity at unit level.
- Problem: Find a production plan that maximizes profit using only the available resources.

Formulation

Find $x = (x_1, \ldots, x_j, \ldots, x_n)$ to solve:

$$\max c \cdot x \text{ subject to } Ax \leq b, x \geq 0.$$

- The objective function just adds up the profits earned from operating each of the activities at level x.
- If you operate activity 1 at level x_1, then you earn $c_1 x_1$ from it.
- Total profit comes from adding the profit of each of the activities.
- The resources constraints state that the production plan does not use up more of any resource than is available.

Continued

- If you follow the production plan x, then

$$\sum_{j=1}^{n} a_{ij}x_j$$

is the amount of the ith resources consumed.

- You need to have

$$\sum_{j=1}^{n} a_{ij}x_j \le b_i$$

for all $i = 1, \ldots, m$.

- The non-negativity constraint states that you cannot operate a production process at a negative level.

What do we know?

- When feasible?
  If resources are available in non-negative quantities ($b \geq 0$).
  In this case, $x = 0$ is an element of the feasible set.
- The LP will have a solution unless it is unbounded.
- In order for the LP to be unbounded, you would have to be able to produce something in arbitrarily large amounts (that is, one of the components of x would be able to grow to infinity).
- Rule this out by assuming every entry in the maxtrix A is strictly positive.

Dual

The mathematical form of the dual is

$$\min b \cdot y \text{ subject to } yA \geq c, y \geq 0.$$

- Some tries to "buyout" the firm.
- It offers to pay y_i per unit for raw material i.
- It tries to buy all raw materials at minimum cost.
- That is, it seeks to find y to minimize $b \cdot y$.
- Prices must be attractive enough to convince firm to sell out.
- What does this mean?

Constraints

- Buyer will pay you y_i for each unit of the ith resource.
- Prices will be high enough so that the inputs of every production process can be sold for at least as much as the output can be sold.
- For each $j = 1, \ldots, n$,

$$\sum_{i=1}^{m} a_{ij} y_i \geq c_j.$$

AND THIS IS THE DUAL

Set prices (of basic resources) $y = (y_1, \ldots, y_i, \ldots, y_m)$ to minimize what the buyout artist pays to acquire the resources $(y \cdot b)$ subject to the constraint that the buyout artist always pays at least as much for the resources as you could get from transforming the resources into a final product.
This should sound like diet problem/pill problem.