

# Econ 172A - Slides from Lecture 4

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# Announcements

1. Quiz Thursday, in class. The last fifteen minutes. Short answer. No appliances.
2. Topics: Formulation and Graphing
3. Problems (from web page):
  - 3.1 Problem Set 1 from 2004, 2007, 2008. You are not responsible for Excel. This means that you can't solve problems from 2004 and Question 2 from 2007 (but the questions should make sense and could provide useful intuition).
  - 3.2 Midterms I: 2004 (2), 2007 (2, 3), 2008 (1, 3)
  - 3.3 Final: 2004 (2)
  - 3.4 Fall 2010, Quiz 1

# DUALITY

Start with an LP written in the form:

$$\max c \cdot x \text{ subject to } Ax \leq b, x \geq 0.$$

Now called **Primal**.

Vital statistics:

1.  $n$  variables (components of  $x$ )
2. and  $m$  constraints
3.  $c$  is  $n$ -dimensional
4.  $b$  is  $m$ -dimensional
5.  $A$  is a matrix with  $m$  rows and  $n$  columns.
6. Given data:  $b$ ,  $c$ , and  $A$ .

## New Problem

$$\min b \cdot y \text{ subject to } yA \geq c, y \geq 0.$$

called **Dual**.

Vital statistics:

1.  $m$  variables (components of  $y$ )
2. and  $n$  constraints
3. Given data:  $b$ ,  $c$ , and  $A$ .



# The Duality Theorem

## Theorem

*If problem (P) has a solution  $x^*$ , then problem (D) also has a solution (call it  $y^*$ ). Furthermore, the values of the problems are equal:  $c \cdot x^* = b \cdot y^*$ . If problem (P) is unbounded, then problem (D) is not feasible.*

*Similarly, if problem (D) has a solution  $y^*$ , then problem (P) also has a solution (call it  $x^*$ ). Furthermore, the values of the problems are equal:  $c \cdot x^* = b \cdot y^*$ . If problem (D) is unbounded, then problem (P) is not feasible.*

The Duality Theorem states that the problems (P) and (D) are intimately related. One way to think about the relationship is to create a table of possibilities.

P \ D	unbounded	has solution	not feasible
unbounded	no	no	possible
has solution	no	same values	no
not feasible	possible	no	possible

Comments:

- ▶ Two random LPs. All nine boxes possible.
- ▶ Minimum Number of boxes possible: 3.
- ▶ With Primal and Dual: 4.

## Another Way

P \ D	feasible	not feasible
feasible	both have solutions; values equal	P unbounded
not feasible	D unbounded	possible

This table shows that if you know that both problems are feasible, then you know that neither problem is unbounded or, equivalently, that both have solutions. More than that, the values of the solutions are equal.

# Intuition for Duality Theorem

- ▶ Duality Theorem requires a mathematical proof.
- ▶ I won't give it, but
- ▶ A **feasible value** for (P) is a number  $v$  with that property that there is some  $x$  such that  $Ax \leq b$  and  $x \geq 0$ , such that  $v = c \cdot x$ .
- ▶ A feasible value for (D) is analogous.

- ▶ Claim: any feasible value for (P) is less than or equal to any feasible value for (D). In symbols:

if  $Ax \leq b, x \geq 0, yA \geq c$  and  $y \geq 0$ , then  $c \cdot x \leq b \cdot y$ .

- ▶ Proof of Claim:  $Ax \leq b$  on both sides by  $y$  and multiply  $yA \geq c$  on both sides by  $x$  to get

$$c \cdot x \leq yAx \leq b \cdot y.$$

and so

$$c \cdot x \leq b \cdot y.$$

# Interpretation

$$c \cdot x \leq b \cdot y.$$

- ▶ (P) feasible implies the value of (D) is bounded below.
- ▶ So (D), a minimization problem, cannot be unbounded.
- ▶ (D) is feasible implies (P) cannot be unbounded.
- ▶ The Duality Theorem goes on to say two more things.
  1. If one problem has a solution, then the other one does.
  2. If the problems have solutions, then the values are equal.

## Using the Duality Theorem

- ▶ Conclude that since both primal and dual are feasible, both must have solutions.

Example: Diet Problem and its Dual are “obviously” feasible.

- ▶ Identify whether a problem is unbounded or infeasible.

Consider the following pair of problems:

$$\begin{array}{rllll} \max & x_1 & + & x_2 & \\ \text{subject to} & -3x_1 & + & 2x_2 & \leq -1 \\ & x_1 & - & x_2 & \leq 2 \\ & & & x & \geq 0 \end{array}$$

The dual is:

$$\begin{array}{rllll} \min & -y_1 & + & 2y_2 & \\ \text{subject to} & -3y_1 & + & y_2 & \geq 1 \\ & 2y_1 & - & y_2 & \geq 1 \\ & & & y & \geq 0 \end{array}$$

1. You could graph each of these problems and easily determine their solution.
2. You could solve them using the simplex algorithm (either by hand or with Excel).
3. But you could use common sense and duality.
  - 3.1 Primal is feasible (  $(x_1, x_2) = (1, 0)$  satisfies the constraints. )
  - 3.2 Dual is infeasible.(Add the two resource constraints in the dual together you get:  $-y_1 \geq 2$ . This inequality implies that  $y_1 \leq -2$ , which is inconsistent with the non-negativity constraint.)
4. Conclude Primal is unbounded.

## Repeat

The insight is that the Duality Theorem allows you to infer something that may not be obvious. There are three kinds of inference.

1. Observe that both primal and dual are feasible. Conclude: both have solution.
2. Observe that primal is feasible and dual is not. Conclude: primal is unbounded.
3. Observe that primal is unbounded. Conclude: dual is infeasible.

(The second and third observations remain true if you interchange the words primal and dual.)

# Complementary Slackness

- ▶ Variables in one problem are **complementary** to constraints in the other. That is, for each variable in the Primal, there is a constraint in the Dual. For each variable in the Dual, there is a constraint in the Primal.
- ▶ **Complementary Slackness** refers to a relationship between the slackness in a primal constraint and the slackness (positivity) of the associated dual variable.

## Theorem (Complementary Slackness)

Assume problem (P) has a solution  $x^*$  and problem (D) has a solution  $y^*$ .

1. If  $x_j^* > 0$ , then the  $j$ th constraint in (D) is binding.
2. If the  $j$ th constraint in (D) is not binding, then  $x_j^* = 0$ .
3. If  $y_i^* > 0$ , then the  $i$ th constraint in (P) is binding.
4. If the  $i$ th constraint in (P) is not binding, then  $y_i^* = 0$ .

## Interpretation

- ▶ If a Primal variable is positive, then the associated dual constraint must be binding.
- ▶ If a Primal constraint fails to bind, then the associated dual variable must be zero.
- ▶ The result says that you cannot have slack in two associated places at the same time (primal variable, dual constraint) or (primal constraint, dual variable).
- ▶ It **is** possible for a primal constraint to be binding while the associated dual variable is equal to zero (that is, no slack in two places), but it is not possible for the primal constraint to have slack (to be non-binding) and the associated dual variable be positive.
- ▶ While I listed four statements in the theorem, there really are only two. The second two statements have precisely the same content as the first two statements, they just switch around the roles of primal and dual.

# Uses

1. Interpretation of Dual Variables.
2. Makes it easier to solve dual knowing solution to primal.

## Why the Complementary Slackness Condition is True

Earlier I showed that if  $x$  is feasible for (P) and  $y$  is feasible for (D), then

$$c \cdot x \leq yAx \leq b \cdot y.$$

Furthermore, if  $x^*$  solves (P) and  $y^*$  solves (D), then the first and last terms are equal, so both inequalities must really be equations:

$$c \cdot x^* = y^*Ax^* = b \cdot y^*.$$

- ▶ Write the first equation as  $(y^*A - c) \cdot x^* = 0$ .
- ▶ Or  $\sum_{j=1}^n (y^*A - c)_j x_j^* = 0$  where I use  $(c - y^*A)_j$  to represent the  $j$ th component of  $y^*A$ .
- ▶ We know that  $(y^*A - c)_j$  is a nonnegative number (this follows because  $y^*$  is feasible for the Dual) and  $x_j^*$  is nonnegative.
- ▶ When we multiply them, we must get a non-negative number.

- ▶ The only way that you can add up a bunch of non-negative numbers and get zero is if each one of the non-negative numbers is zero.
- ▶ Hence, for each  $j = 1, \dots, n$ ,  $(c - y^*A)_j x_j^* = 0$ .
- ▶ This expression says that when you multiply two numbers  $((y^*A - c)_j$  and  $x_j^*)$  you get zero.
- ▶ This can only happen if at least one of the numbers is zero.
- ▶ This is precisely what the complementary slackness condition says.
- ▶ If  $x_j^* > 0$ , then the  $j$ th dual constraint binds (that is,  $(y^*A - c)_j = 0$ ) and if the  $j$ th dual constraint does not bind (that is,  $(y^*A - c)_j > 0$ , then  $x_j^* = 0$ ).
- ▶ You can derive the other CS conditions by applying the same kind of reasoning to the equation  $y^*Ax^* = b \cdot y^*$ .

# Complementary Slackness and Interpretation of Dual

Remember Diet Problem.

- ▶ Suppose in solution to diet problem you buy positive quantity of first food.
- ▶ What does this mean?
- ▶ CS says: First dual constraint must bind (at solution).
- ▶ What does this mean?
- ▶ The first food is no more expensive than the pills you must buy to replace it.
- ▶ What if the second dual constraint is not binding?
- ▶ The second food is more expensive than the nutrients it contains.
- ▶ So you don't buy the second food.

- ▶ What if the first primal constraint does not bind?
- ▶ The solution to the diet problem supplies too much of the first nutrient.
- ▶ So that nutrient should be free.
- ▶ What if a nutrient has a positive price.
- ▶ Then you satisfy no more than the minimum requirement of that nutrient when you solve the diet problem.

It is possible for a food's cost to be equal to its nutritional content and still not be consumed. In this case there is no slack in the constraint "pills are cheaper than food" and in the nonnegativity constraint.

Algebraically: If the product of two numbers is zero, it is possible that both numbers are zero.

# Equal Values

The complementary slackness conditions guarantee that the values of the primal and dual are the same.

In the diet problem, the pill seller guarantees that pills are no more expensive than food. CS guarantees that when you solve the problem, pills cost exactly the same as food for those foods that you actually buy.