Econ 172A - Slides from Lecture 3

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Announcements

1. Quiz Next Thursday, in class. Fifteen minutes. Short answer. No appliances.
2. Topics: Formulation and Graphing
3. Problems (from web page):
    3.1 Problem Set 1 from 2004, 2007, 2008.
    3.2 Midterms I: 2004 (1, 2), 2007 (2, 3), 2008 (1, 3)
    3.3 Final: 2004 (2)
    3.4 2010, Quiz 1.

Graphing Linear Inequalities in the Plane

1. Two variable LPs can be solved graphically.
2. You need to know two things:
   - Graph linear inequalities in the plane (you probably did this in high school)
   - Figure out the relationship between these points and the objective function.

Graph Line

For example: $2x_1 + x_2 = 2$,

$(x_1, x_2) = (1, 0)$ and $(x_1, x_2) = (0, 2)$ are on the line. Connect the dots.

Graph Halfplane

The inequality $2x_1 + x_2 \geq 2$, consists of all of the points above and to the right of the straight line.

- In general: inequalities are satisfied by points on one side of the line.
- To determine which set consists of the point that satisfies the inequality, test by checking an arbitrary point not on the line.
- For example, $(x_1, x_2) = (0, 0)$ does not satisfy the inequality $2x_1 + x_2 \geq 2$.
- Hence the set of points that satisfies the inequality consists of the points on the side of the line $2x_1 + x_2 = 2$ that does not contain $(0, 0)$.

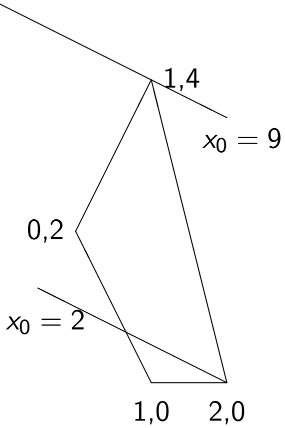
Many Constraints

For example, the set determined by the five inequalities

$$\begin{array}{rcrcr} 2x_1 & + & x_2 & \geq & 2 \\ -2x_1 & + & x_2 & \leq & 2 \\ 4x_1 & + & x_2 & \leq & 8 \\ & & x & \geq & 0. \end{array}$$

This is region bounded by the quadrilateral pictured. (The four corners are $(0,2)$, $(1,0)$, $(2,0)$, and $(1,4)$.)

Picture



Comments

- 5 inequalities? The first three lines describe one inequality each. The fourth line describes two: $x_1 \geq 0$ and $x_2 \geq 0$.
- If you have five inequalities, you would expect the feasible set of have five sides.
- This set has only four because the constraint that $x_1 \geq 0$ is redundant.
- If you satisfy the other four constraints, then you automatically satisfy $x_1 \geq 0$.
- Flawed intuition: you should have as many variables as equations to have a system that makes sense. Not true here. Reasons:
  1. Inequalities not equations.
  2. You want "large" feasible set.

Corners

In the example, the feasible set has four corners. These corners are determined by the intersection of pairs of constraints, solved as equations. That is, $(0, 2)$ is the solution to

$$\begin{array}{rcrcl} 2x_1 & + & x_2 & = & 2 \\ -2x_1 & + & x_2 & = & 2, \end{array}$$

$(1, 4)$ is the solution to

$$\begin{array}{rcrcl} -2x_1 & + & x_2 & = & 2 \\ 4x_1 & + & x_2 & = & 8, \end{array}$$

$(1, 0)$ is the solution to $2x_1 + x_2 = 2$ and $x_2 = 0$, and $(2, 0)$ is the solution to $4x_1 + x_2 = 8$ and $x_2 = 0$.

More Generally

- The feasible region of a linear programming problem has corners determined by solving subsets of the constraints as equations.
- Once you have these corners, you get the feasible set by connecting the dots and identifying the region that satisfies all of the constraints.
- The feasible set may be empty. Replace the constraint that $x_1 \geq 0$ with one that said that $x_1 \leq -1$.
- The feasible set may be unbounded. That is, it may go out forever in one or more directions. (Having no constraints is perfectly ok.)
- The only way to have a problem that has an unbounded solution is to have an unbounded feasible set.

Graphical Solutions

To solve LP graphically:

1. Graph feasible set. If empty, stop (problem is not feasible). If non empty,
2. Find solution if it exists. (Solution must exist if feasible set is bounded. It might exist otherwise.)
3. Graph a **level set** of the objective function.
   Level set of f:
   $$\{x : f(x) = c\}.$$
   Level sets of linear functions in the plane are lines.
4. Adjust level set so that it intersects feasible set.
5. Increase value of objective function until the greatest possible intersection.

SUMMARY

1. Graph feasible set. If feasible set is empty, then stop. The problem is infeasible. Otherwise continue.
2. Graph a level set of the objective function.
3. Shift the level set (parallel movement) until it intersects the feasible region.
4. Continue to shift the level set until it reaches the maximum value the intersects the feasible region.

Comments

1. Follow the same steps for a minimization problem, taking care to move the objective function in the opposite direction.
2. You know which direction increases the objective function value by drawing two level sets and comparing (the direction of increase never changes).
3. In the example, the level set $x_1 + 2x_2 = 9$ lies above and to the right of the level set $x_1 + 2x_2 = 2$; you always increase the objective function (in this example) by moving up and to the right.

UNBOUNDED

- If your feasible set is unbounded, then it may be that the linear programming problem is unbounded.
- You will be able to see this graphically if level sets for arbitrarily large values of the objective function continue to intersect the feasible region.

Big Ideas

- *If a linear programming problem has a solution, then it has a solution at a "corner" of the feasible set.*
- Therefore, if you want to solve a LP and you know the corners of the feasible set, then all you need to do is plug the corners into the objective function and pick the best one.
- (In the example, the feasible set has only four corners: $(0, 2)$, $(1, 0)$, $(2, 0)$, and $(1, 4)$.)
- The standard algorithm for solving LPs (**the simplex algorithm** uses this idea).
- Linear programming problems may have solutions that are not at corners. For example, if $x_0 = 4x_1 + x_2$, then all points on the segment connecting $(1, 4)$ to $(2, 0)$ solve the LP.

MORE

- If a linear programming problem has two solutions, then everything on the segment connecting the two solutions is also a solution.
- If the feasible set is unbounded, then testing only at the corners of the feasible region will tell you the solution to the problem *if* the LP has a solution. The problem may be unbounded.

Unbounded Problems

Omit the constraint that $4x_1 + x_2 \leq 8$.

The solution to $\min x_1 + x_2$ subject to:

$$\begin{array}{rcrcl} 2x_1 & + & x_2 & \geq & 2 \\ -2x_1 & + & x_2 & \leq & 2 \\ & & x & \geq & 0 \end{array}$$

is $(1, 0)$.

No solution to $\max x_1 + x_2$ subject to these constraints.

The Form of Answers

If I ask you to solve a linear programming problem graphically, then you should give one of three answers.

1. If the problem is not feasible, then you demonstrate that the graph of the constraint set is empty (and state that the problem is infeasible).
2. If the problem has a solution, then you should state the solution (or solutions, if there is more than one); show a level set of the objective function that intersects the feasible region at the solution; and show that if you increase the objective function the corresponding level set would no longer intersect the feasible region.
3. If the problem is unbounded, then you should show how you can find a feasible point that makes the objective function larger (or, in the case of a minimization problem, smaller) than any arbitrarily chosen value M.

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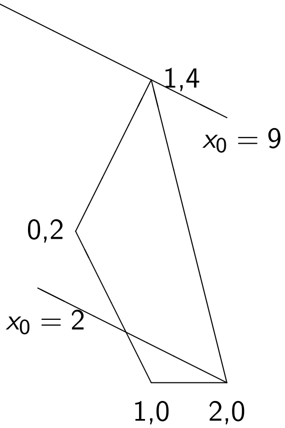

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Topics for Quiz 1 end here

DUALITY

Start with an LP written in the form:

$$\max c \cdot x \text{ subject to } Ax \leq b, x \geq 0.$$

Now called **Primal**.
Vital statistics:

1. n variables (components of x)
2. and m constraints
3. c is $n-$dimensional
4. b is $m-$dimensional
5. A is a matrix with m rows and n columns.
6. Given data: b, c, and A.

New Problem

$$\min b \cdot y \text{ subject to } yA \geq c, y \geq 0.$$

called **Dual**.
Vital statistics:

1. m variables (components of y)
2. and n constraints
3. Given data: b, c, and A.

Why?

1. What is the point? Answer: Later.
2. Do all LPs have duals? Answer: Yes
3. Can you keep taking the dual? Answer: Dual of Dual is Dual.

Constructing Duals

1. Switch objective from max to min.
2. Constraints switch from $\leq$ to $\geq$.
3. The number of constraints changes from m to n.
4. The number of variables changes from n to m.
5. The objective function coefficients switch roles with the right-hand side constants.

Example

$$\begin{array}{lrcrcrcrcr}
\max & 2x_1 & + & 4x_2 & + & 3x_3 & + & x_4 & & \\
\text{subject to} & 3x_1 & + & x_2 & + & x_3 & + & 4x_4 & \leq & 12 \\
 & x_1 & - & 3x_2 & + & 2x_3 & + & 3x_4 & \leq & 7 \\
 & 2x_1 & + & x_2 & + & 3x_3 & - & x_4 & \leq & 10 \\
 & & & & & & & x & \geq & 0
\end{array}$$

Dual:

$$\begin{array}{lrcrcrcr}
\min & 12y_1 & + & 7y_2 & + & 10y_3 & & \\
\text{subject to} & 3y_1 & + & y_2 & + & 2y_3 & \geq & 2 \\
 & y_1 & - & 3y_2 & + & y_3 & \geq & 4 \\
 & y_1 & + & 2y_2 & + & 3y_3 & \geq & 3 \\
 & 4y_1 & + & 3y_2 & - & y_3 & \geq & 1 \\
 & & & & & y & \geq & 0
\end{array}$$

The Duality Theorem

Theorem

If problem (P) has a solution x^*, then problem (D) also has a solution (call it y^*). Furthermore, the values of the problems are equal: $c \cdot x^* = b \cdot y^*$. If problem (P) is unbounded, then problem (D) is not feasible.
Similarly, if problem (D) has a solution y^*, then problem (P) also has a solution (call it x^*). Furthermore, the values of the problems are equal: $c \cdot x^* = b \cdot y^*$. If problem (D) is unbounded, then problem (P) is not feasible.

The Duality Theorem states that the problems (P) and (D) are intimately related. One way to think about the relationship is to create a table of possibilities.