Econ 172A - Slides from Lecture 19

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December 6, 2012

Announcements

- Review session: Saturday 12:30-2 in 107 Solis.
- Final Examination (Monday December 10, 3-6)
  1. Comprehensive
  2. Emphasis on second half
  3. Rule of thumb: one week of lecture counts 10% of final grade. Hence one week of material not covered on previous quizzes or exams will be roughly 20% of final grade.
  4. Old finals will be a good (but imperfect) guide.
  5. Review this year's quizzes and midterm.
- Revised seating for final posted.
- Complete CAPEs, please (you have until Monday morning)

Stable Matching

1. N boys, and N girls.
2. Everyone wants to be matched with (one) member of the opposite sex. That is, everyone is marriage to exactly one person of the opposite sex.
3. People have preferences (each boy can order the girls, best, second best, third best, . . . ; each girl can order the boys).
4. Want arrangement to be stable. Stability means that there does not exist a boy and a girl who are not marriage who prefer to be married to each other than their current partner. (That is, stability requires that if B_i is not married to G_j then either B_i prefers his wife to G_j or G_j prefers her husband to B_i (or both).

Compare to Assignment Problems

1. No costs associated with each pair (c_{ij} if B_i marries G_j).
2. No single objective to maximize (or minimize).
3. c_{ij} is cost of assignment, it does not attribute to each person in a marriage individually. (One could create a model in which b_{ij} is what B_i gets if married to G_j and g_{ij} is what G_j gets if married to B_i and then $c_{ij} = b_{ij} + g_{ij}$.)
4. The stable matching model cares only about the order people put on potential mates (not absolute differences).

Stability Again

A match between boys and girls is stable if there does not exist a boy-girl pair (all them Ben and Jen) such that:

1. Ben is not paired with Jen.
2. Ben prefers Jen to his mate.
3. Jen prefers Ben to her mate.

Suppose that you could find a pair that satisfied the three properties. The matching is unstable in the following sense. Ben can approach Jen and suggest that she dump her current partner in favor of him. Ben hopes Jen will accept (because he likes her better than his current partner). Jen will accept (because she likes Ben better than her current partner).

Observations

1. If a matching isn't stable, you will expect divorces.
2. Stability does not prevent someone from being stuck with his least preferred choice: Suppose that I am every girl's least favorite boy. If I am matched to my least favorite girl, then the first two conditions above hold between me and any girl I'm not matched with, but the third condition fails. When I request that someone break up with their mate to hook up with me, she'll turn me down.

Central Questions

1. Do stable matchings exist?
2. How do you find them?
3. What are their properties?

Answers: yes, through an algorithm, not enough time.

An Algorithm

Step 1. Each boy “proposes” to the favorite girl on his list.

Step 2. Each girl who receives at least one proposal, “dates” the boy she prefers among those who propose; “rejects” the rest. Girls with no proposals do nothing.

Step 3. If no boy is rejected, stop. You have a stable matching between girls and their current dates. Otherwise, rejected boys cross the name of the girl who rejected them off their list and then propose to the favorite among those remaining. Boys who are “dating” repeat their proposal.

Step 4. Return to Step 2.

Variation

The algorithm works just as well if the boys propose one at a time and as soon as a girl receives multiple offers, she rejects her less favorite boy.

In the previous algorithm, the boys all propose at the same time. In the variation, the boys propose one at a time and keep proposing until some girl has two offers. When she does, she rejects the one she likes less and the rejected boy proposes to the next girl on his list.

More precisely, assume that the boys are ordered, 1, . . . , N. Initially, no boys or girls have partners. A boy who proposes to a girl has that girl as a partner until the girl rejects him. When a boy is rejected, he has no partner until he makes another proposal.

Alternative Algorithm

Step 1. The first boy "proposes" to the favorite girl on his list.

Step 2. The lowest numbered boy without a partner "proposes" to the favorite girl remaining on his list.

Step 3 If all girls have partners, stop. You have a stable marriage. Otherwise go to Step 4.

Step 4a. If a girl has more than one partner, then she rejects the boy she prefers less (and keeps the other as a partner). The rejected boy crosses this girl off his list. Return to Step 2.

Step 4b. If no girl has more than one partner, return to Step 2.

Animation

This link takes you to the webpage that demonstrates the algorithm:

Stable Matching Algorithm

Analysis

We need to verify:

- The algorithm is well defined (it is possible for everyone to follow the steps);
- When the algorithm stops it provides a matching;
- The algorithm stops in finite time;
- When the algorithm stops, it stops at a stable matching.

Details

1. If a girl ever receives a proposal, then she has a date in each subsequent period (because she never rejects all of the boys who propose and the boy she is dating cannot propose to anyone else).
2. The algorithm stops when each girl is dating exactly one boy (so that no boy is rejected).
3. It follows that the algorithm must stop before any boy is rejected by every girl. (If a boy is rejected by all but one girl, then he proposes to the last girl on his list. At that time, the other girls must all be dating someone – the boy they preferred to the rejected boy or someone even better. So when the last girl gets a proposal, the algorithm stops.)
4. The algorithm is well defined because no boy is rejected by all of the girls (so there is always someone left to propose to) and because preferences are strict (so that there is always a favorite).

Continued

5 The algorithm ends at a matching because after the final round of proposals, no girl could have received more than one proposal (because that would lead to a rejection) and each boy must be dating at least one girl.

6 The algorithm ends in a finite number of steps because in each round (until the last one) at least one boy is rejected. No boy can be rejected more than $N-1$ times. Since there are N boys, in no more than $N(N-1)$ rounds the process must stop.

Algorithm gives stability

1. Suppose Ben is not married to Jen, but Ben prefers Jen to his mate Gwen.
2. If Ben prefers Jen to Gwen, then according to the algorithm, he proposed to Jen before he proposed to Gwen.
3. So Jen must have rejected him.
4. She would only have done that if some boy she preferred had also proposed to her.
5. Gwen must end up mated with someone at least as good (according to her preferences) as the guy who was better than Ben.
6. Hence she prefers her mate to Ben and the match must be stable.

Property

Definition: A stable matching that is **boy optimal** if, among all other stable matchings, it is the one that is unanimously preferred by the boys.

Claim: Algorithm provides a boy optimal matching.

Discussion

1. That any matching is boy optimal is surprising.
   - Different boys certainly have different preferences over **all** matchings. (For example, if two boys have the same favorite girl, then they disagree about which is the best matching.)
   - What is surprising is that If you limit attention to stable matchings, the boys all have the same interest.
   - Also, the algorithm seems to give a lot of power to the girl. They can, after all, select who to reject.
2. In fact, it is the power to make offers, however, that is more valuable.

Official Statement

The Gale-Shapley Algorithm supplies a boy-optimal stable match.

1. Say that boy B_i is an eligible mate for G_j if there exists some stable matching in which they are married.
2. Goal: to show that the algorithm assigns to each boy his most preferred eligible partner.
3. Argue to contradiction.

More

4. Assume some boy is paired with someone other than his best eligible partner.
5. Hence this boy is rejected by an eligible partner.
6. Let Lloyd be the first such boy, and let Amy be first valid partner that rejects him.
7. When Lloyd is rejected, Amy must have available a boy, say David, whom she prefers to Lloyd.
8. Let M be the stable matching in which Amy and Lloyd are mates.
9. Let Beth be David's partner in the matching M.

Conclusion of argument

10. Claim: M is not a stable matching.
    If Claim holds we have a contraction.

Proof of Claim

- By assumption, David was not rejected by any eligible partner at the point when Lloyd is rejected by Amy (since Lloyd is first to be rejected by an eligible partner).
- David prefers Amy to Beth.
- But Amy prefers David to Lloyd.

Continuation

- The matching M cannot be stable.

Related Claim

The Gale-Shapley Algorithm finds the girl-pessimal stable matching. (Each girl is married to worst eligible partner.)

1. Suppose algorithm matches Amy to David, but David is not the worst eligible partner for Amy.
2. There exists stable matching M in which Amy is paired with Lloyd, whom she likes less than David.
3. Let Beth be David's partner in M.
4. David prefers Amy to Beth (because the algorithm selects a man optimal matching.
5. Hence Amy and David prefer each other to their mate in M.
6. M therefore cannot be a stable matching.

Lying

Suppose that boys and girls know that matches will be made according to the algorithm. Is it in their best interest to make and respond to proposals honestly?
Claim (hard to prove): Iif all of the girls behave honestly (rejecting boys they don't prefer in favor of those that they do prefer), then it is in the best interest of boys to behave honestly (after all, this leads to their most preferred stable match).
Claim (easier to see): Girls might gain by dating someone who is not their best current option with the expectation that they'll eventually get someone they like even better.

Example

BOY	1	2	3
Adam	Amy	Beth	Cara
Bill	Beth	Amy	Cara
Carl	Amy	Beth	Cara

Boys' Preferences

GIRL	1	2	3
Amy	Bill	Adam	Carl
Beth	Adam	Bill	Carl
Cara	Adam	Bill	Carl

Girls' Preferences

Proof

- The Algorithm arrives at the match: Adam-Amy; Bill-Beth; Carl-Cara if girls are honest.
- Instead assume that Amy is dishonest in the first round.
- In the first round she gets proposals from Adam and Carl.
- Suppose that she rejects Adam (who she likes better than Carl).
- Then Adam will propose to Beth, who will accept him and reject Bill.
- Bill then approaches Amy, who is now able to dump Carl.
- Eventually, Cara gets stuck with Carl and the match: Adam-Beth; Bill-Amy; and Carl-Cara results.
- This match is stable, but it is better for Amy and Beth (and worse for Adam and Bill) than the Gale-Shapley match.

Quote

Most mathematicians at one time or another have probably found themselves in the position of trying to refute the notion that they are people with "a head for figures," or that they "know a lot of formulas." At such times it may be convenient to have an illustration at hand to show that mathematics need not be concerned with figures, either numerical or geometrical. For this purpose we recommend the statement and proof of our Theorem 1. The argument is carried out not in mathematical symbols but in ordinary English; there are no obscure or technical terms. Knowledge of calculus is not presupposed. In fact, one hardly needs to know how to count. Yet any mathematician will immediately recognize the argument as mathematical, while people without mathematical training will probably find difficulty in following the argument, though not because of unfamiliarity with the subject matter.