Econ 172A - Slides from Lecture 18

Joel Sobel

December 4, 2012

Announcements

- 8-10 this evening (December 4) in York Hall 2262
- I'll run a review session here (Solis 107) from 12:30-2 on Saturday.
- Quiz 4 answers posted after class.
- Up to date grade information to follow (next page).
- Final Examination (Monday December 10, 3-6)
  1. Comprehensive
  2. Emphasis on second half
  3. Rule of thumb: one week of lecture counts 10% of final grade. Hence one week of material not covered on previous quizzes or exams will be roughly 20% of final grade.
  4. Old finals will be a good (but imperfect) guide.
  5. Review this year's quizzes and midterm.
- Revised seating for final posted after class.

Grading Update

1. Quiz 4: 42 possible (6 points per question), median 30
2. Points from Quizzes and Midterm: 200 possible.
3. Computation: Midterm + .75(Best Two Quizzes)

Decile	Cumulative Points
Top 10%	174
Top 20%	166
Top 30%	160
Top 40%	155
Top 50%	146
Top 60%	140
Top 70%	131
Top 80%	124
Top 90%	109

Example

Consider the assignment problem with the costs given in the array below.

	1	2	3	4
A	10	7	8	2
B	1	5	6	3
C	2	10	3	9
D	4	3	2	3

This array describes an assignment problem with four people (labeled A, B, C, and D) and four jobs (1, 2, 3, 4). The first person has a cost 10 if assigned to the first job; a cost 7 if assigned to the second job; etc. The goal is to assign people to jobs in a way that minimizes total cost.

Simplification

I can subtract the "fixed cost" for the other three people (rows) so that there is guaranteed to be at least one zero in each row. I obtain:

	1	2	3	4
A	8	5	6	0
B	0	4	5	2
C	0	8	1	7
D	2	1	0	1

Columns

- I can subtract a constant from any column.
- Take the second column.
- It says that no matter who you assign to the second job, it will cost at least 1.
- Treat the 1 as a fixed cost and subtract it.
- Since it cannot be avoided it does not influence your solution (it does influence the value of the solution).

Once you make this reduction you get:

	1	2	3	4
A	8	4	6	0
B	0	3	5	2
C	0	7	1	7
D	2	0	0	1

Next Step

- The last table is simpler that the original one.
- There is a zero in every row and in every column.
- All of the entries are non-negative.
- Since you want to find an assignment that minimizes total cost, it would be ideal if you could find an assignment that only pairs people to jobs when the associated cost is zero.
- Keep this in mind: The goal of the computation is to write the table in a way that is equivalent (has the same solution) as the original problem and has a zero-cost assignment.
- I have just finished the step in which you reduce the costs so that there is at least one zero in every row and every column.
- The example demonstrates that this is not enough.

No Solution Yet

- Try to find zero-cost assignment.
- You must assign A to 4 (the only zero in the row for A is in the 4 column).
- You must assign B to 1.
- The only way to get a zero cost from C is to assign it to 1 as well.
- Impossible: I have already assigned B to 1.

- Intuition suggests that you should do the next best thing: assign C to job 3 (at the cost 1) and then D to 2.
- This yields the solution to the problem (A to 4; B to 1; C to 3; D to 2). It is not, however, an algorithm.
- We made the final assignments by guessing. (This is the solution. I argued that it is impossible to solve the problem at cost zero, but then demonstrated that it is possible to solve the problem at the next best cost, one.)

Algorithm

- The trick is to eliminate the zeros in the table and then try to reduce the remaining values.
- The key is to find a transformation that can find a cheaper assignment without changing the essence of the problem.

Repeat

	1	2	3	4
A	$\underline{8}$	$\underline{4}$	$\underline{6}$	$\underline{0}$
B	0	3	5	2
C	0	7	1	7
D	$\underline{2}$	$\underline{0}$	$\underline{0}$	$\underline{1}$

- I crossed out two rows and one column.
- Doing so "covers up" all of the zeros.
- Now look at the uncovered cells and find the smallest number (it turns out to be one).
- If I subtracted one from each cell in the entire matrix, then I would leave the basic problem unchanged (that is, I would not change the optimal assignment) and I would "create" a new low cost route (C to 3).
- However, some entries (covered by lines) would become negative.
- This is bad news because if there are negative entries, there is no guaranteed that a zero-cost assignment really minimizes cost.
- So reverse the process by adding the same constant you subtracted from every entry (1) to each row and column with a line through it.

Doing so creates this cost matrix:

	1	2	3	4
A	9	4	6	0
B	0	2	4	1
C	0	6	0	6
D	3	0	0	1

- The table is that it again is non-negative.
- Using this matrix it is possible to make another minimum cost assignment.
- In fact, using this table, we can come up with an optimal assignment with cost zero.
- It agrees with our intuition (A to 4; B to 1; C to 3; D to 2).
- You can go back to the original matrix of costs to figure out what the total cost is: $9 = 2 + 1 + 3 + 3$.

Mechanically:

1. Subtract the minimum number from each zero to leave one zero element in each row.
2. Subtract the minimum number from each column to leave one zero element in each column.
3. Find the minimum number of lines that cross out all of the zeros in the table.

4. From all of the entries that are not crossed out, find the minimum number (it should be positive). If the minimum is zero, then you haven't crossed out enough entries. If all of the entries are crossed out, then you already should be able to find a zero cost assignment.
5. Subtract the number that you found in Step 4 from all of the entries that haven't been crossed out. Do not change the entry in any cell that has one line through it. Add the number to those entries that have two lines through it.
6. Return to Step 1.

Discussion

- The first two steps are simple. They make the problem more transparent.
- The third and fourth steps are general versions of the first two steps.
- What you do in these steps is redistribute the costs in a way that does not change the solution.
- Step 3 is the mysterious step.
- Cross out all of the zeros in the table using the minimum number of lines.
- I recommend that you do this by finding the row or column that has the most zeros; cross that one out.
- Next, cross out the row or column that has the most remaining uncrossed zeros. (There may be more than one way to do this.)
- Continue until you are done.

- In Step 5 you do two things.
- First, you subtract the number you found in Step 4 from **every** element of the table.
- This does not change the solution.
- It does, however, create negative numbers.

- Hence you must do something to restore non-negativity in the cost table (otherwise you cannot apply the rule that you want to find a zero-cost assignment to solve the problem).
- You do this by adding the constant back to every row or column that you draw a line through.
- When all is done, you are left with a table that satisfies the properties in Step 5.
- All entries that are not "lined" go down.
- The ones that have one line through them stay the same (go down and then go up by the same amount).
- The ones that have two lines (none will have three) go up (they go down, but then they go up twice).

Finishing

You are done when you reach a stage in which you can find a zero-cost assignment.

- I won't provide a general procedure for doing this.
- It is natural to start by looking to see if any row or column has exactly one zero in it.
- If it does, you must include the assignment corresponding to that cell.
- Do so, cross out the corresponding row and column, and solve the remaining (smaller) problem.
- If each row and column contains at least two zeros, make one assignment using an arbitrary row and column (with a zero cell) and continue.
- The problems that I ask you to solve will be small enough to solve by trial and error.

Loose End

- I have not demonstrated that the algorithm must give you a solution in a finite number of steps.
- The basic idea is that each step lowers the cost of your assignment.
- Verifying this requires a small argument.

Another Example

	1	2	3	4	5
A	81	14	36	40	31
B	20	31	25	26	81
C	30	87	19	70	65
D	23	56	60	18	45
E	12	15	18	21	100

I will first subtract the minimum element in each row:

	1	2	3	4	5
A	67	0	22	26	17
B	0	11	5	6	61
C	11	68	0	51	46
D	5	38	42	0	27
E	0	3	6	9	88

Next, I subtract the minimum element from each column (only the fifth column has no zero in it).

	1	2	3	4	5
A	$\vert\underline{67}$	$\underline{0}$	$\vert\underline{22}$	$\vert\underline{26}$	$\underline{0}$
B	$\vert 0$	11	$\vert 5$	$\vert 6$	44
C	$\vert 11$	68	$\vert 0$	$\vert 51$	29
D	$\vert 5$	38	$\vert 42$	$\vert 0$	10
E	$\vert 0$	3	$\vert 6$	$\vert 9$	71

This array does not permit a zero-cost solution (both 2 and 5 must be matched with A). Hence we need to change it.

	1	2	3	4	5
A	70	0	25	29	0
B	0	8	5	6	41
C	11	65	0	51	26
D	5	35	42	0	7
E	0	0	6	9	68

- From this array we can find a zero-cost assignment.
- The solution is A to 5; B to 1; C to 3; D to 4; and E to 2.
- Using the costs from the original table, the cost of this plan is:

$$31 + 20 + 19 + 18 + 15 = 103.$$

Maximization

Suppose that we had exactly the same information as we had in the previous example:

	1	2	3	4	5
A	81	14	36	40	31
B	20	31	25	26	81
C	30	87	19	70	65
D	23	56	60	18	45
E	12	15	18	21	100

but these numbers were benefits and not costs. Then our objective would be to maximize the value of the assignment. How do we do this?

Multiply Entries by -1

	1	2	3	4	5
A	-81	-14	-36	-40	-31
B	-20	-31	-25	-26	-81
C	-30	-87	-19	-70	-65
D	-23	-56	-60	-18	-45
E	-12	-15	-18	-21	-100

This changes the optimization problem from a maximum to a minumum.

Add a Constant to make entries positive

You cannot apply to algorithm to the previous table because it has negative entries. So you can add a large enough constant to make all entries positive. (You can do this all at once, or row-by-row). I just added 100 to all entries:

	1	2	3	4	5
A	19	86	64	60	69
B	80	69	75	74	19
C	70	13	11	30	35
D	77	44	40	82	55
E	88	85	82	79	0

Now you can solve this problem using the algorithm. (Subtract constants so that there is a zero in each row and column; look for zero-cost matching; stop if you find one; cross out zeros and subtract and add if you don't; continue.)

Answer

Clearly, the assignment that maximizes the benefits won't be the same as the one that minimizes costs. (That is, the answer to the maximum problem won't be the same as the answer to the minimization problem.)

I completed the computation and found that the solution to the maximization problem involves: A-1, B-4, C-2, D-3, E-5. The benefit (values from the original table) is

$81 + 26 + 87 + 60 + 100 = 354$

Transportation Problems are Assignment Problems

In the transportation problem, there are n supply centers. Supply center i has supply S_i. There are m demand centers. Demand center has demand D_j. The cost of sending one unit from supply center i to demand center j is c_{ij}. The objective is to meet all of the demands at minimum cost.
This is a generalization of the assignment problem in two ways.

1. First, in the assignment problem, all supplies and demands are equal ($D_j = S_i = 1$).
2. Second, in the assignment problem the total demand is equal to the total supply.

Adjusting for the differences

To solve the first problem (supplies and demands greater than one), imagine a problem in which demand center j is divided into D_j separate demand centers, each with demand one and Supply center i is divided into S_i separate supply centers, each with supply one.

Extra Supply

To solve the second problem, assume $\sum_{j=1}^{m} D_j \leq \sum_{i=1}^{n} S_i$ (otherwise the problem is not feasible). Create a new demand center, 0, call it "the dump."

- Set $D_0 = \sum_{i=1}^{n} S_i - \sum_{j=1}^{m} D_j$
- Set $c_{i0} = 0$ all i

Interpretation: You can throw away excess supply at zero cost.

Summary

With these two changes, any feasible transportation problem can be thought of as an assignment problem and solved using the assignment problem algorithm.