Econ 172A - Slides from Lecture 17

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Announcements

- Network problems: 2007: Homework 3, #1; Final: #5,6
  2008: Homework 3 # 1, 2; Final: #4.
- Coming attractions (administrative):
  1. Revised seating for the final.
  2. Update on grades (including all quizzes and midterm)
- Coming attractions (substantive):
  1. Finish Max Flow-Min Cut
  2. Assignment Problems
  3. Matching Algorithm (not in notes)

The Assignment Problem

- N positions.
- N people.
- Must assign each person to exactly one position.
- Must assign each position to exactly one persion.
- a_{ij} is the value of assigning position j to person i. [or c_{ij} is the cost.]
- Object: Find the assignment that maximizes total value.

Formulation

If we let x_{ij} be equal to 1 if player i is assigned to position j and equal to zero otherwise, then the problem is to find x_{ij} to solve:

$$\max \sum_{i=1}^{n} \sum_{j=1}^{n} x_{ij} a_{ij}$$

subject to

$$\sum_{i=1}^{n} x_{ij} = 1 \text{ for } j = 1, \ldots, n$$

and

$$\sum_{j=1}^{n} x_{ij} = 1 \text{ for } i = 1, \ldots, n.$$

Also, the variables x_{ij} must take on the values 0 or 1.
(If instead you are given costs, then your objective is to minimize.)

Applications

- Women to men.
- Workers to jobs.
- Doctors to residency programs.
- Students to schools.
- Kidneys to patients.

The Hungarian Method

- The assignment problem is a linear programming problem (with the additional constraint that the variables take on the values zero and one).
- Generally, the additional constraint makes the problem quite difficult.
- However, the assignment problem has the property that when you solve the problem ignoring the integer constraints you still get integer solutions.
- This means that the simplex algorithm solves assignment problems.
- Assignment problems have so much special structure that there are simpler algorithms available for solving them.
- I will describe one of the algorithms, called the Hungarian method.

Example

Consider the assignment problem with the costs given in the array below.

	1	2	3	4
A	10	7	8	2
B	1	5	6	3
C	2	10	3	9
D	4	3	2	3

This array describes an assignment problem with four people (labeled A, B, C, and D) and four jobs (1, 2, 3, 4). The first person has a cost 10 if assigned to the first job; a cost 7 if assigned to the second job; etc. The goal is to assign people to jobs in a way that minimizes total cost.

- Observation: You can subtract a constant from any row or column without changing the solution to the problem.
- Take the first row (the costs associated with A).
- All of these numbers are at least two.
- Since you must assign person A to some job, you must pay at least two no matter what.
- If you'd like, think of that as a fixed cost and further costs as variable costs depending on the job assigned to the first person.
- Hence if I reduce all of the entries in the first row by two, I do not change the optimal assignment (I lower the total cost by two).

Doing so leaves this table:

	1	2	3	4
A	8	5	6	0
B	1	5	6	3
C	2	10	3	9
D	4	3	2	3

The solution to the problem described by the second table is exactly the same as the solution to the first problem.

Continuing in this way I can subtract the "fixed cost" for the other three people (rows) so that there is guaranteed to be at least one zero in each row. I obtain:

	1	2	3	4
A	8	5	6	0
B	0	4	5	2
C	0	8	1	7
D	2	1	0	1

Columns

- I can subtract a constant from any column.
- Take the second column.
- It says that no matter who you assign to the second job, it will cost at least 1.
- Treat the 1 as a fixed cost and subtract it.
- Since it cannot be avoided it does not influence your solution (it does influence the value of the solution).

Once you make this reduction you get:

	1	2	3	4
A	8	4	6	0
B	0	3	5	2
C	0	7	1	7
D	2	0	0	1

Next Step

- The last table is simpler that the original one.
- There is a zero in every row and in every column.
- All of the entries are non-negative.
- Since you want to find an assignment that minimizes total cost, it would be ideal if you could find an assignment that only pairs people to jobs when the associated cost is zero.
- Keep this in mind: The goal of the computation is to write the table in a way that is equivalent (has the same solution) as the original problem and has a zero-cost assignment.
- I have just finished the step in which you reduce the costs so that there is at least one zero in every row and every column.
- The example demonstrates that this is not enough.

No Solution Yet

- Try to find zero-cost assignment.
- You must assign A to 4 (the only zero in the row for A is in the 4 column).
- You must assign B to 1.
- The only way to get a zero cost from C is to assign it to 1 as well.
- Impossible: I have already assigned B to 1.

- Intuition suggests that you should do the next best thing: assign C to job 3 (at the cost 1) and then D to 2.
- This yields the solution to the problem (A to 4; B to 1; C to 3; D to 2). It is not, however, an algorithm.
- We made the final assignments by guessing. (This is the solution. I argued that it is impossible to solve the problem at cost zero, but then demonstrated that it is possible to solve the problem at the next best cost, one.)