Econ 172A - Slides from Lecture 16

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Announcements

- Network problems: 2007: Homework 3, #1; Final: #5,6
  2008: Homework 3 # 1, 2; Final: #4.
- Today is last day for midterm regrade requests.
- Quiz 4 on Thursday. Quiz 4 from 2010 is a useful guide, but material on shortest route and minimum spanning tree are also "fair game."
- Coming attractions (administrative):
  1. Revised seating for the final.
  2. Update on grades (including all quizzes and midterm)
- Coming attractions (substantive):
  1. Finish Max Flow-Min Cut
  2. Assignment Problems
  3. Matching Algorithm (not in notes)

Application

- m jobs and n people.
- Each person can be assigned to do only one job and is only able to do a particular subset of the jobs.
- Question: What is the maximum number of possible jobs that can be done?

Set up

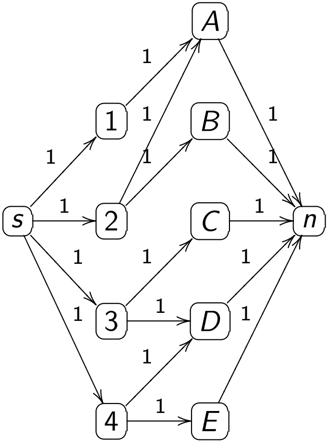
- Set up a network in which there is an arc with capacity one connecting the source to each of the n nodes (one for each person).
- and an arc with capacity one connecting each person to each job that the person can do,
- and an arc with capacity one connecting each job to the sink.

Example

- Four people and five jobs.
- The first person can do only job A.
- The second person can do either job A or job B.
- The third person can do either job C or job D.
- The fourth person can do either job D or job E.

Solution

- You can get 4 jobs done.
- Not so easy when there are many jobs and people.



Cuts

There is also a way to prove that your answer is correct.

- Define a **cut** to be a partition of the nodes into two sets, once set containing (s) and the other set containing (n).
- The **cut capacity** is the total capacity from the part of the cut containing the source to the part of the cut containing the sink.
- The capacity of any cut is greater than or equal to any feasible flow.
- Hence finding any cut can give you an upper bound on the maximum flow.
- For example, consider the cut $\{(s),(1),(2),(3),(4),(A),(B),(C),(D),(E)\} \cup \{(n)\}$.
- It has capacity 5, so the maximum flow cannot exceed 5.

Max Flow is equal to Min Cut Capacity

- Since the capacity of any cut is greater than or equal to the maximum flow, if you can ever find a cut that has capacity equal to a feasible flow, then you know that you have solved the problem.
- This is exactly what happens at the end of the algorithm.
- Whenever you reach a point where you can no longer find a flow augmenting path, you are able to generate a cut by taking as one set the set of labeled nodes and the other set the rest.
- For the final diagram of the example, the cut $\{(s)\} \cup \{(1),(2),(3),(4),(A),(B),(C),(D),(E),(n)\}$ has capacity four.
- A cut created in this fashion has capacity equal to the maximum flow (otherwise you would have been able to label another node).

Amazing Claim

The general fact that the minimum capacity cut is equal to the maximum flow is a consequence of the duality theorem of linear programming.

Animation

Minimum Cut

Another Application

- Given a number of warehouses, each with a fixed supply of something.
- You are given a number of markets, each with a fixed demand for that thing.
- You are given the capacity of the various shipping routes (from warehouse i to market j).
- Your problem is to determine whether it is possible to meet the given demand.

Example

For example, the information could be:

	Markets				Supplies
Warehouse	1	2	3	4	
1	30	10	0	40	20
2	0	0	10	50	20
3	20	10	40	5	100
Demands	20	20	60	20	

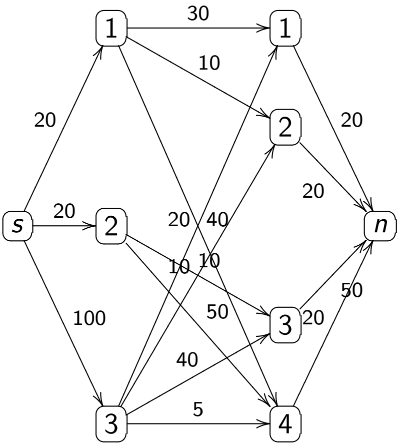
Interpretation

- There are three warehouses.
- Reading down the last column we see that the supply available at each one (so warehouse one has supply 20).
- There are four markets.
- Reading across the last row we see the demand available at each one (so the demand in Market 3 is 60).
- The numbers in the table describe what can be shipped directly from one warehouse to a market (so you can ship as many as forty units from the third warehouse to the third market).

Comment

- Construct a network from this information.
- From the sink there is an arc to a node for each warehouse.
- The capacity of the arc is the supply at the warehouse.
- From each warehouse node there is an arc to a node fro each market.
- The capacity of the arc is the maximum feasible flow to that market.
- From each market node there is an arc to the sink with capacity equal to the demand in that market.

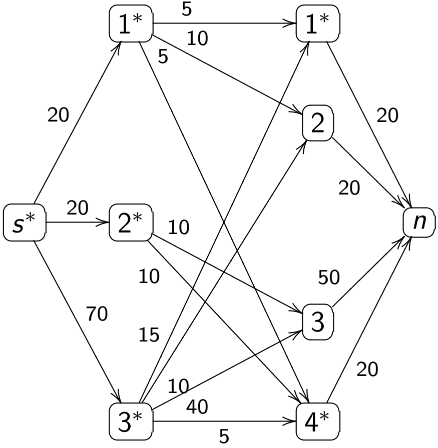
For the example, therefore, the relevant network is:



Solution

- Solve the associated maximum flow problem.
- Figure out the most that can be shipped from the warehouses to the markets.
- In order to answer the question: Can you meet the given demand? You just check to see whether the maximum flow is equal to the total demand.

The next table illustrates the associated minimum cut (the nodes with asterisks are in the part of the cut containing the source). Since it yields a flow of 110 while the total demand is 120, it is not possible to meet all of the demands.


1*
5
1*
10
5
20
2
20
20
s*
20
2*
10
n
10
50
70
3
15
20
10
40
3*
4*
5

Linear Programming Formulation

Number the nodes so that source is node 0 and sink in node $N+1$ (and the other nodes are $1, \ldots, N$. Let x_{ij} be equal to the flow from Node i to Node j. The problem is to find x_{ij} to solve:

$$\max \sum_{i=0}^{N} x_{in}$$

subject to

$$\sum_{i=0}^{N} x_{ik} = \sum_{j=1}^{N+1} x_{kj} \text{ for } k = 1, \ldots, N$$

and

$$0 \leq x_{ij} \leq c_{ij}$$